Week #8 - Optimization and Newton's Method

Section 4.5

From "Calculus, Single Variable" by Hughes-Hallett, Gleason, McCallum et. al.
Copyright 2005 by John Wiley & Sons, Inc.
This material is used by permission of John Wiley & Sons, Inc.

SUGGESTED PROBLEMS

1. *The bending moment M of a beam, supported at one end, at a distance x from the support is given by*

$$M = \frac{1}{2}wLx - \frac{1}{2}wx^2$$

where L is the length of the beam, and w is the uniform load per unit length. Find the point on the beam where the moment is greatest.

We look for critical points of M:

$$\frac{dM}{dx} = \frac{1}{2}wL - wx$$

Now $dM/dx = 0$ when $x = L/2$. At this point $d^2M/dx^2 = -w$ so this point is a local maximum. The graph of $M(x)$ is a parabola opening downward, so the local maximum is also the global maximum.

7. *The efficiency of a screw, E, is given by*

$$E = \frac{\theta - \mu\theta^2}{\mu + \theta}$$

where θ is the angle of pitch of the thread and μ is the coefficient of friction of the material, a (positive) constant. What value of θ maximizes E?

Differentiating, to look for critical points:

$$\frac{dE}{d\theta} = \frac{\mu + \theta(1 - 2\mu\theta) - (\theta - \mu\theta^2)}{(\mu + \theta)^2}$$

$$= \frac{\mu(1 - 2\mu\theta - \theta^2)}{(\mu + \theta)^2}$$

$dE/d\theta = 0$ when $\theta = -\mu \pm \sqrt{1 + \mu^2}$. Since $\theta > 0$, the only possible critical point is when $\theta = -\mu + \sqrt{1 + \mu^2}$. Slopes go from positive (at $\theta = 0$ to negative after the critical point, so it is a local maximum. Because of the positive slopes to the left and negative slopes to the right of the critical the local maximum must be the global maximum.

11. *Figure 4.69 shows the curves $y = \sqrt{x}$, $x = 9$, $y = 0$, and a rectangle with its sides parallel to the axes and its left end at $x = a$. Find the dimensions of the rectangle having the maximum possible area.*
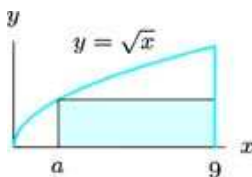

Figure 4.69

We wish to choose a to maximize the area of the rectangle with corners at $(a, \sqrt{a})$ and $(9, \sqrt{a})$. The area of this rectangle will be given by the formula

$$R = h \cdot l = \sqrt{a}(9 - a) = 9a^{1/2} - a^{3/2}$$

We are restricted to $0 \leq a \leq 9$. To maximize this area, we set $dR/da = 0$, and then check that the resulting area is greater than the area if $a = 0$ or $a = 9$. Since $R = 0$ if $a = 0$ or $a = 9$, all we need to do is to find where $dR/da = 0$:

$$\frac{dR}{da} = \frac{9}{2}a^{-1/2} - \frac{3}{2}a^{1/2} = 0$$
$$\frac{9}{2\sqrt{a}} = \frac{3\sqrt{a}}{2}$$
$$18 = 6a$$
$$a = 3$$

Thus the dimensions of the maximal rectangle are 6 by $\sqrt{3}$.

17. *If you have 100 feet of fencing and want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose?*

Let w and l be the width and length, respectively, of the rectangular area you wish to enclose. Then

$$\text{length} = w + w + l = 100 \text{ feet}$$
$$l = 100 - 2w$$

Using that relationship, we can simplify the area to be a function of only w.

$$\text{Area} = w \cdot l = w(100 - 2w) = 100w - 2w^2$$

To maximize the area, we look for the critical point of the area. If we let the area be representing $A(w)$,

2

$$A'(w) = 100 - 4w$$

Setting the derivative equal to zero, $4w = 100$ or $w = 25$. Because the area function is a parabola, opening downwards (negative coefficient of w^2 in $A(w)$), we know that this represents the global maximum of the area.

Thus the maximum area of $A(25) = w \cdot (100 - 2w) = 25 \times 50 = 1250$ square feet is achieved when $w = 25$ feet.

34. *To get the best view of the Statue of Liberty in Figure 4.76, you should be at the position where θ is a maximum. If the statue stands 92 meters high, including the pedestal, which is 46 meters high, how far from the base should you be?*

   Hint: [Find a formula for θ in terms of your distance from the base. Use this function to maximize θ, noting that $0 \leq \theta \leq \pi/2$.]

Figure 4.76

We want to maximize the viewing angle, which is $\theta = \theta_1 - \theta_2$.
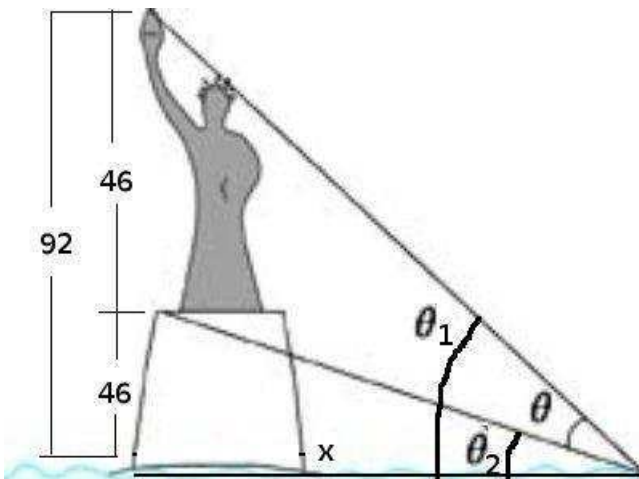
Optimizing this requires some careful trig work. If the distance from the boat to the base

of the statue is x, then

$$\tan(\theta_1) = \frac{92}{x} \qquad\qquad \text{so } \theta_1 = \arctan\left(\frac{92}{x}\right)$$

$$\tan(\theta_2) = \frac{46}{x} \qquad\qquad \text{so } \theta_1 = \arctan\left(\frac{46}{x}\right)$$

Then

$$\theta = \theta_1 - \theta_2 = \arctan\left(\frac{92}{x}\right) - \arctan\left(\frac{46}{x}\right) \text{ for } x > 0$$

We look for critical points of the function by computing $d\theta/dx$, remembering that $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$:

$$\begin{aligned}
\frac{d\theta}{dx} &= \frac{1}{1+(92/x)^2}\left(\frac{-92}{x^2}\right) - \frac{1}{1+(46/x)^2}\left(\frac{-46}{x^2}\right) \\
&= \frac{-92}{x^2+92^2} + \frac{46}{x^2+46^2} \\
&= \frac{-92(x^2+46^2) + 46(x^2+92^2)}{(x^2+92^2)(x^2+46^2)} \\
&= \frac{46(4232-x^2)}{(x^2+92^2)(x^2+46^2)}
\end{aligned}$$

Setting this derivative equal to zero,

$$\begin{aligned}
0 &= \frac{46(4232-x^2)}{(x^2+92^2)(x^2+46^2)} \\
x^2 &= 4232 \\
x &= \pm\sqrt{4232} \approx \pm 65.05
\end{aligned}$$

Since $x > 0$, the critical point is $x = \sqrt{4232} \approx 65.1$ meters. To verify that this is indeed where θ attains a maximum, we not that $d\theta/dx > 0$ for $0 < x < \sqrt{4232}$ and $d\theta/dx < 0$ for $x > \sqrt{4232}$, and so is a local minimum. Given the sign of the slopes on $0 < x < \infty$, this must also be the global maximum.

The best distance to stay view the statue is roughly 65 meters away from the base.

QUIZ PREPARATION PROBLEMS

4

5. *In a chemical reaction, substance A combines with substance B to form substance Y. At the start of the reaction, the quantity of A present is a grams, and the quantity of B present is b grams. Assume $a < b$. At time t seconds after the start of the reaction, the quantity of Y present is y grams. For certain types of reactions, the rate of the reaction, in grams/sec, is given by*

$$Rate \;=\; k(a - y)(b - y)$$

   (a) For what values of y is the rate nonnegative? Graph the rate against y.

   (b) Use your graph to find the value of y at which the rate of the reaction is fastest.

   (a) If we expect the rate to be nonnegative, we must have $0 \le y \le a$ and $0 \le y \le b$. Since we assume $a < b$, we restrict y to $0 \le y \le a$.

   In fact, the expression for the rate is nonnegative for y greater than b, but these values of y are not meaningful for the reaction.
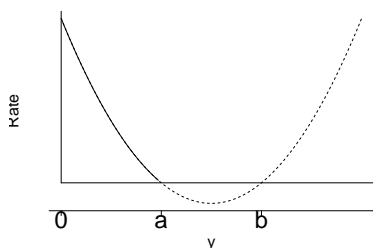
   (b) From the graph, we see that the maximum rate occurs when $y = 0$; that is, at the start of the reaction.

18. *A rectangular beam is cut from a cylindrical log of radius 30 cm. The strength of a beam of width w and height h is proportional to wh^2. (See Figure 4.70.) Find the width and height of the beam of maximum strength.*
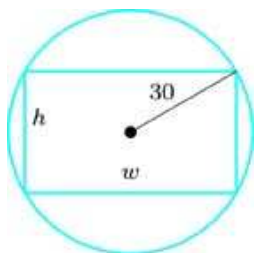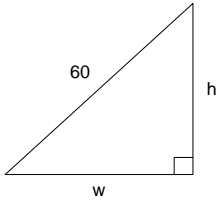

   Figure 4.70

   Since the beam diagonal length must always be 60, h and w must fit into the right angle triangle shown below:

Thus $w^2 + h^2 = 60^2$, or $h^2 = 3600 - w^2$.

We want to optimize the strength:

$$S = kwh^2$$
$$\text{replacing } h^2 \text{ with } 3600 - w^2\text{:} \quad = kw(3600 - w^2)$$
$$= k(3600w - w^3)$$

Differentiating, setting $S'(w) = 0$,

$$S'(w) = k(3600 - 3w^2)$$
$$\text{Setting S' } = 0\text{: } 0 = 3600 - 3w^2$$
$$w^2 = 1200$$
$$w = \pm\sqrt{1200} \approx \pm 34.64$$

Only the positive solution is feasible for this problem, so $w = +\sqrt{1200}$ is the critical point of interest.

To show this point is a global maximum for possible ranges of w, $0 \le w \le 60$, we note that $S(0) = 0$ and $S(60) = 0$. Thus, a beam with dimensions $w \approx 34.64$ cm and $l = \sqrt{3600 - w^2} \approx 48.99$ cm will have the optimal strength.

19. *A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden. She will use shrubs costing $25 per foot along three sides and fencing costing $10 per foot along the fourth side. Find the minimum total cost.*
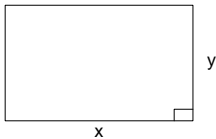
The total area is $xy = 3000$, so $y = 3000/x$. Suppose that the left and right edges and the lower edge have the shrubs, while the top edge has the fencing. Then the total cost function would be

$$C = 25(y + x + y) + 10x = 50y + 35x$$
$$C(x) = 50\frac{3000}{x} + 35x$$
$$= \frac{150,000}{x} + 35x$$

Finding the critical points,

$$C'(x) = \frac{-150,000}{x^2} + 35$$
$$\text{Setting } C' = 0: \ 0 = \frac{-150,000}{x^2} + 35$$
$$35x^2 = 150,000$$
$$x = \pm\sqrt{\frac{150,000}{35}} = \pm\sqrt{\frac{30,000}{7}} \approx \pm65.5 \text{ feet}$$

Only the positive root is feasible, as we are only interested in $x > 0$, so $x \approx +65.5$ is the only critical point of interest.

To show this is a global minimum on the domain of the problem, we compare this with

- $\lim_{x \to 0} C(x) = +\infty$
- $\lim_{x \to \infty} C(x) = +\infty$

Thus the critical point must be a global minimum, so $x \approx 65.5$ feet and $y = 3000/x \approx 48.5$ feet give the dimensions of the yard with the lowest fencing/shrub cost, $C(65.5) \approx \$4583$.

22. *A light is suspended at a height h above the floor. (See Figure 4.71.) The illumination at the point P is inversely proportional to the square of the distance from the point P to the light and directly proportional to the cosine of the angle θ. How far from the floor should the light be to maximize the illumination at the point P?*
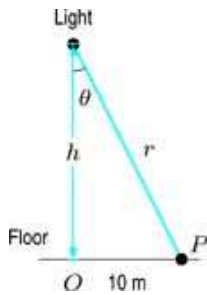

Figure 4.71

Since $r^2 = y^2 + 10^2$ and $\cos\theta = h/r = h/\sqrt{h^2 + 10^2}$, we have

$$I = k\frac{\cos\theta}{r^2}$$

$$= \frac{kh}{(h^2 + 10^2)^{3/2}}$$

To find the height at which $I(h)$ is maximized, we look for critical points of $I(h)$:

$$\frac{dI}{dh} = \frac{k(h^2 + 100)^{3/2} - (\frac{3}{2}(h^2 + 100)^{1/2}2h)(kh)}{(h^2 + 100)^3}$$

$$\text{Take out common factors: } = \frac{k(h^2 + 100)^{1/2}[(h^2 + 100) - 3h^2]}{(h^2 + 100)^3}$$

$$\text{Cancel } (h^2 + 100)^{1/2} \text{ with denominator: } = k\frac{(h^2 + 100) - 3h^2}{(h^2 + 100)^{5/2}}$$

$$\text{Simplify numerator: } = k\frac{100 - 2h^2}{(h^2 + 100)^{5/2}}$$

Setting this equal to zero to identify critical points, and solving for h:

$$0 = k\frac{100 - 2h^2}{(h^2 + 100)^{5/2}}$$

$$100 = 2h^2$$

$$h = \sqrt{50}$$

Since $dI/dh > 0$ for $0 \leq h < \sqrt{50}$, and $dI/dh < 0$ for $h > \sqrt{50}$, we know that the light intensity is a maximum when $h = \sqrt{50}$.

31. *A bird such as a starling feeds worms to its young. To collect worms, the bird flies to a site where worms are to be found, picks up several in its beak, and flies back to its nest. The loading curve in Figure 4.73 shows how the number of worms (the load) a starling collects depends on the time it has been searching for them.5 The curve is concave down because the bird can pick up worms more efficiently when its beak is empty; when its beak is partly full, the bird becomes much less efficient. The traveling time (from nest to site and back) is represented by the distance PO in Figure 4.73. The bird wants to maximize the rate at which it brings worms to the nest, where*

$$Rate\ worms\ arrive\ = \frac{Load}{Traveling\ time\ +\ Searching\ time}$$

(a) *Draw a line in Figure 4.73 whose slope is this rate.*

(b) *Using the graph, estimate the load which maximizes this rate.*

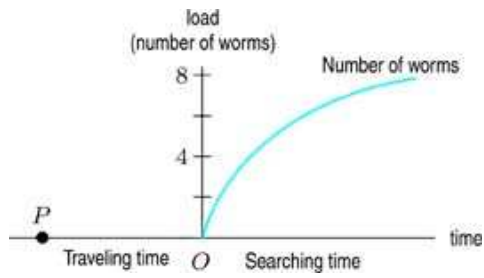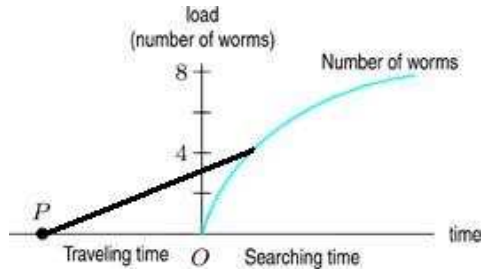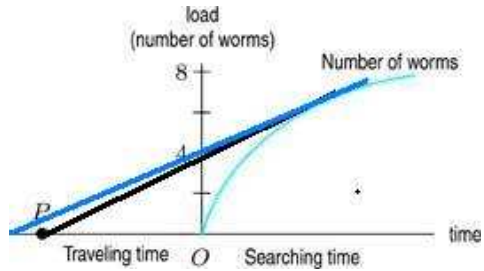(c) *If the traveling time is increased, does the optimal load increase or decrease? Why?*

Figure 4.73

(a) A possible line is shown on the graph below. Its slope is (load collected) / (total travel time), or the rate at which worms arrive back at the nest.

(b) The slope of the line is maximized when the line is tangent to the loading curve, which happens with the black line on the graph below.

(c) If the traveling time is increase, the point P moves to the left, the blue line on the graph below would show the new optimal slope. The contact point is higher, so the bird would be more heavily loaded, but would have a net lower efficiency, as the blue line has a lower slope than the earlier black line.