Week #9 - The Integral

Section 5.1

From "Calculus, Single Variable" by Hughes-Hallett, Gleason, McCallum et. al.

SUGGESTED PROBLEMS

7. *For time, t, in hours, $0 \leq t \leq 1$, a bug is crawling at a velocity, v, in meters/hour given by*

$$v = \frac{1}{1+t}$$

Use $\Delta t = 0.2$ to estimate the distance that the bug crawls during this hour. Find an overestimate and an underestimate. Then average the two to get a new estimate.

Using $\Delta t = 0.2$, we need the following values

t	$v(t) = \frac{1}{1+t}$	
0.0	1/(1+0)	= 1
0.2	1/(1+0.2)	= 5/6
0.4	1/(1+0.4)	= 5/7
0.6	1/(1+0.6)	= 5/8
0.8	1/(1+0.8)	= 5/9
1.0	1/(1+1.0)	= 5/10

The left-hand estimate is

$$\begin{aligned} &v(0)\Delta t + v(0.2)\Delta t + \ldots + v(0.8)\Delta t \\ &= (0.2)[1 + 5/6 + 5/7 + 5/8 + 5/9] \\ &\approx 0.74563 \text{ meters} \end{aligned}$$

The right-hand estimate is

$$\begin{aligned} &v(0.2)\Delta t + v(0.4)\Delta t + \ldots + v(1.0)\Delta t \\ &= (0.2)[5/6 + 5/7 + 5/8 + 5/9 + 5/10] \\ &\approx 0.64563 \text{ meters} \end{aligned}$$

Because the bug is moving slower and slower (see the table), the left-hand sum is an overestimate of the distance traveled, while the right-hand sum is an underestimate.

The average of the two estimates is ≈ 0.6956 meters. This a reasonable estimate of the distance the bug has traveled.

9. *The velocity of a car is $f(t) = 5t$ meters/sec. Use a graph of $f(t)$ to find the exact distance traveled by the car, in meters, from $t = 0$ to $t = 10$ seconds.*

A graph of the simple linear function is shown below, along with the area (marked in grey) beneath the graph.

The area under the rectangle represents the distance traveled, and it is

$$\frac{1}{2} \cdot 10 \cdot 50 = 250$$

The car travels 250 meters in the first 10 seconds.

16. *Roger runs a marathon. His friend Jeff rides behind him on a bicycle and clocks his speed every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. Jeff's data follow:*

Time since start (min)	*0*	*15*	*30*	*45*	*60*	*75*	*90*
Speed (mph)	*12*	*11*	*10*	*10*	*8*	*7*	*0*

(c) *How often would Jeff have needed to measure Roger's speed in order to find lower and upper estimates within 0.1 mile of the actual distance he ran?*

(a,b) See solutions to Quiz Prep problems

(c) The difference between Roger's pace at the beginning and the end of his run is 12 mph. If the time between the measurements is h, then the difference between the upper and lower estimates is $12h$. We want $12h < 0.1$, so

$$h < \frac{0.1}{12} \approx 0.0083 \text{ hours} = 30 \text{ seconds}$$

Thus Jeff would have to measure Roger's pace every 30 seconds.

QUIZ PREPARATION PROBLEMS

15. *A student is speeding down Route 11 in his fancy red Porsche when his radar system warns him of an obstacle 400 feet ahead. He immediately applies the brakes, starts to slow down, and spots a skunk in the road directly ahead of him. The black box in the Porsche records the car's speed every two seconds, producing the following table. The speed decreases throughout the 10 seconds it takes to stop, although not necessarily at a uniform rate.*

Time since brakes applied (sec)	*0*	*2*	*4*	*6*	*8*	*10*
Speed (ft/sec)	*100*	*80*	*50*	*25*	*10*	*0*

(a) *What is your best estimate of the total distance the student's car traveled before coming to rest?*

(b) *Which one of the following statements can you justify from the information given?*

(i). The car stopped before getting to the skunk.

(ii). The black box data is inconclusive. The skunk may or may not have been hit.

(iii). The skunk was hit by the car.

(a) Our best estimate would be the average of the left- and right-hand sums. Since the interval between measurements is $\Delta t = 2$ seconds, the left-hand sum is

$$\begin{aligned} &\Delta t \cdot [v(0) + v(2) + v(2) + v(6) + v(8)] \\ &= 2 \cdot [100 + 80 + 50 + 25 + 10] \\ &= 530 \text{ feet} \end{aligned}$$

The right-hand sum is

$$\begin{aligned} &\Delta t \cdot [v(2) + v(2) + v(6) + v(8) + v(10)] \\ &= 2 \cdot [80 + 50 + 25 + 10 + 0] \\ &= 330 \text{ feet} \end{aligned}$$

Since the driver was braking continuously, the velocity should have been decreasing the whole time. This means that the left-hand sum is an overestimate of the stopping distance while the right-hand sum is an underestimate.

A more accurate estimate would be to average the two numbers: 430 feet.

(b) All we can be sure of is that the distance traveled lies between the upper and lower estimates calculate above. In other words, all the black-box data tells us is that the car traveled between 330 and 530 feet before stopping. As a result, we can't be completely sure if it hit the skunk, which has 400 feet away when the braking began.

16. *Roger runs a marathon. His friend Jeff rides behind him on a bicycle and clocks his speed every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. Jeff's data follow:*

Time since start (min)	*0*	*15*	*30*	*45*	*60*	*75*	*90*
Speed (mph)	*12*	*11*	*10*	*10*	*8*	*7*	*0*

(a) *Assuming that Roger's speed is never increasing, give upper and lower estimates for the distance Roger ran during the first half hour.*

(b) *Give upper and lower estimates for the distance Roger ran in total during the entire hour and a half.*

(a) Note that 15 minutes equals 0.25 hours.

Left-hand estimate = (0.25) [12 + 11] = 5.75 miles.

Right-hand estimate = (0.25) [11+ 10] = 5.25 miles.

The left-hand estimate is an upper estimate, while the right-hand estimate is a lower estimate.

(b) Left-hand estimate = (0.25) [12+11+10+10+8+7] = 14.5 miles.
Right-hand estimate = (0.25) [11+10+10+8+7+0] = 11.5 miles.
Again, the left-hand estimate is an upper estimate, while the right-hand estimate is a lower estimate.

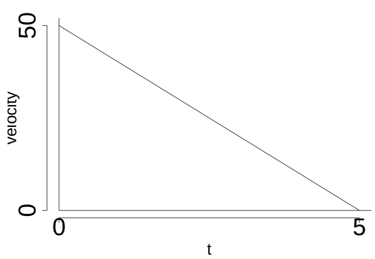
23. *A car initially going 50 ft/sec brakes at a constant rate (constant negative acceleration), coming to a stop in 5 seconds.*

(a) Graph the velocity from $t = 0$ to $t = 5$.

(b) How far does the car travel?

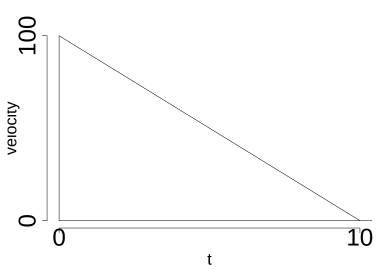
(c) How far does the car travel if its initial velocity is doubled, but it brakes at the same constant rate?

(a) The acceleration is constant, so the velocity graph is linear, through the points ($t = 0$, $v = 50$) and ($t = 5, v = 0$).



(b) The distance traveled is the same as the area under the graph of the velocity. The region is a triangle of base 5 and height 50, so the area is $0.5 \cdot 5 \cdot 50 = 125$. Thus the distance traveled is 125 feet.

(c) The slope of the graph of the velocity function is the same, so the triangular region under it has twice the altitude and twice the base (it takes twice as much time to stop). See the graph below. This scaling produces a triangle that is 4 times larger than the original, so the stopping distance is 4 times longer, or $4 \cdot 125 = 500$ feet.



25. *Two cars start at the same time and travel in the same direction along a straight road. Figure 5.13 gives the velocity, v, of each car as a function of time, t. Which car:*

(a) Attains the larger maximum velocity?

(b) Stops first?

(c) *Travels farther?*

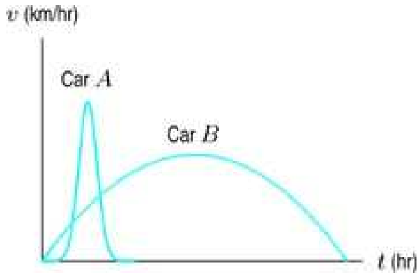


Figure 5.13

(a) Car A has the largest maximum velocity because the peak of car A's velocity curve is higher than the peak of B's.

(b) Car A stops first because the curve representing its velocity hits zero (on the t-axis) first.

(c) Car B travels farther because the area under car B's velocity curve is the larger.

26. *Two cars travel in the same direction along a straight road. Figure 5.14 shows the velocity, v, of each car at time t. Car B starts 2 hours after car A and car B reaches a maximum velocity of 50 km/hr.*

(a) *For approximately how long does each car travel?*

(b) *Estimate car A's maximum velocity.*

(c) *Approximately how far does each car travel?*

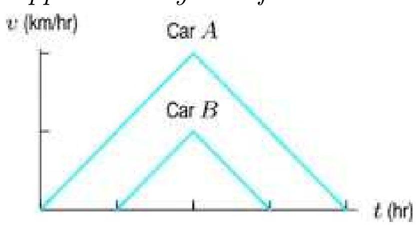


Figure 5.14

(a) Since car B starts at $t = 2$, the tick marks on the horizontal axis (which we assume are equally spaced) are 2 hours apart. Thus car B stops at $t = 6$ and travels for 4 hours.

(b) Car A's maximum velocity is approximately twice that of car B, or 100 km/hr.

(c) The distance traveled is given by the area under the velocity graph. Using the formula for the area of a triangle, the distances are given approximately by

$$\begin{aligned}\text{Car A distance} &= \frac{1}{2}\cdot\text{Base}\cdot\text{Height} = \frac{1}{2}\cdot 8\cdot 100 = 400 \text{ km}\\ \text{Car B distance} &= \frac{1}{2}\cdot\text{Base}\cdot\text{Height} = \frac{1}{2}\cdot 4\cdot 50 = 100 \text{ km}\end{aligned}$$