Week #17 - Differential Equations

Section 11.5

From "Calculus, Single Variable" by Hughes-Hallett, Gleason, McCallum et. al.

SUGGESTED PROBLEMS

1. *Each curve in Figure 11.38 represents the balance in a bank account into which a single deposit was made at time zero. Assuming continuously compounded interest, find:*

   (a) The curve representing the largest initial deposit.

   (b) The curve representing the largest interest rate.

   (c) Two curves representing the same initial deposit.

   (d) Two curves representing the same interest rate.

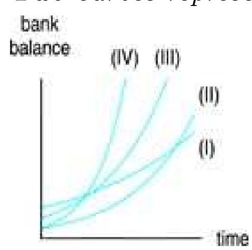

Figure 11.38

   (a) (I) - Initial deposit is the height at time $t = 0$.

   (b) (IV) - grows the most quickly/has steepest slopes for same y values

   (c) (II) and (IV)

   (d) (II) and (III)

2. *The graphs in Figure 11.39 represent the temperature, H (^oC), of four eggs as a function of time, t, in minutes. Match three of the graphs with the descriptions (a)-(c). Write a similar description for the fourth graph, including an interpretation of any intercepts and asymptotes.*

   (a) An egg is taken out of the refrigerator (just above 0C) and put into boiling water.

   (b) Twenty minutes after the egg in part (a) is taken out of the fridge and put into boiling water, the same thing is done with another egg.

   (c) An egg is taken out of the refrigerator at the same time as the egg in part (a) and left to sit on the kitchen table.

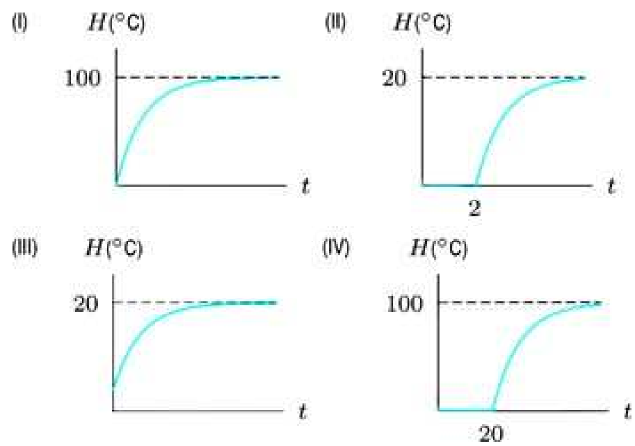

Figure 11.39

(a) (I) - starts at H = 0 at $t = 0$, starts rising towards 100 immediately.

(b) (IV) - The egg stays at 0 degrees for 20 minutes, then rises towards 100 degrees.

(c) (III) - The temperature starts at $H = 0$ at $t = 0$, but immediately begins to rise towards the room temperature, 20^o.

9. *Hydrocodone bitartrate is used as a cough suppressant. After the drug is fully absorbed, the quantity of drug in the body decreases at a rate proportional to the amount left in the body. The half-life of hydrocodone bitartrate in the body is 3.8 hours, and the usual oral dose is 10 mg.*

(a) Write a differential equation for the quantity, Q, of hydrocodone bitartrate in the body at time t, in hours, since the drug was fully absorbed.

(b) Solve the differential equation given in part (a)

(c) Use the half-life to find the constant of proportionality, k.

(d) How much of the 10 mg dose is still in the body after 12 hours?

(a) $\dfrac{dQ}{dt} = -kQ$

(b) The solution will be exponential: $Q = Q_0 e^{-kt}$

(c) If the half-life is 3.8 hours, then

$$\frac{1}{2}Q_0 = Q_0 e^{-k \cdot 3.8}$$
$$k = \frac{-1}{3.8}\ln\left(\frac{1}{2}\right) \approx 0.182$$

(d) If $Q_0 = 10$, $Q(12) = 10e^{-0.182 \cdot 12} \approx 1.126$ mg. So there will be approximately 1.126 mg left in the body after 12 hours.

22. *At 1:00 pm one winter afternoon, there is a power failure at your house in Wisconsin, and your heat does not work without electricity. When the power goes out, it is 68F in your house. At 10:00 pm, it is 57F in the house, and you notice that it is 10F outside.*

(a) *Assuming that the temperature, T, in your home obeys Newton's Law of Cooling, write the differential equation satisfied by T.*

(b) *Solve the differential equation to estimate the temperature in the house when you get up at 7:00 am the next morning. Should you worry about your water pipes freezing?*

(c) *What assumption did you make in part (a) about the temperature outside? Given this (probably incorrect) assumption, would you revise your estimate up or down? Why?*

(a) $\dfrac{dT}{dt} = -k(T - T_{ext})$, where T is the temperature in the house, and T_ext is the temperature outside the house.

(b) If we assume that the exterior temperature is 10 degrees during the whole power outage, we can solve the DE to get the temperature over time:

$$\begin{aligned}
& & T &= Ae^{-kt} + 10 \\
&\text{Let 1 PM be } t = 0\text{:} & 68 &= A + 10 \\
& & A &= 58 \\
&\text{At 10 PM, } t = 9\text{:} & 57 &= 58e^{-9k} + 10 \\
& & k &= \frac{1}{9}\ln\left(\frac{58}{47}\right) \approx 0.023366 \\
&\text{At 7 AM, } t = 18\text{:} & T &= 58e^{-18k} + 10 \\
& & &= 48.06
\end{aligned}$$

At 7 AM, the temperature will have dropped to 48 degrees, or well above freezing at 32 degrees Fahrenheit. There is no risk of the pipers freezing by morning.

(c) We assumed that the temperature outside the house would always be 10 degrees, which is a gross over-simplification. Since we would expect the outside temperature to drop between 10 PM and 7 AM, that would mean the house cools more during that period than our model predicts, so the temperature at 7 AM would likely be lower than 48 degrees.

23. *The radioactive isotope carbon-14 is present in small quantities in all life forms, and it is constantly replenished until the organism dies, after which it decays to stable carbon-12 at a rate proportional to the amount of carbon-14 present, with a half-life of 5730 years. Suppose $C(t)$ is the amount of carbon-14 present at time t.*

(a) *Find the value of the constant k in the differential equation $C' = -kC$.*

(b) *In 1988 three teams of scientists found that the Shroud of Turin, which was reputed to be the burial cloth of Jesus, contained 91% of the amount of carbon-14 contained in freshly made cloth of the same material.4 How old is the Shroud of Turin, according to these data?*

(a) You can use a half-life formula. We'll solve the DE, though, for practice.

$$\begin{aligned} C' &= -kC \\ \text{Has solution} \quad C &= C_0 e^{-kt} \\ \text{Half-life is 5730 years:} \quad \frac{C_0}{2} &= C_0 e^{-k\ 5730} \\ k &= \frac{-\ln(\frac{1}{2})}{5730} \approx 1.2097 \times 10^{-4} \end{aligned}$$

(b) In 1988, the amount of carbon is $0.91C_0$:

$$\begin{aligned} 0.91C_0 &= C_0 e^{-kt} \\ t &= \frac{\ln(0.91)}{-k} \approx 780 \end{aligned}$$

From this calculation, it seems that the cloth was made from living plant material roughly 780 years ago, rather than the 2,000 years claimed by the discoverers.

QUIZ PREPARATION PROBLEMS

3. *A yam is put in a 200C oven and heats up according to the differential equation*

$$\frac{dH}{dt} = -k(H - 200)$$

(a) *If the yam is at 20C when it is put in the oven, solve the differential equation.*

(b) *Find k using the fact that after 30 minutes the temperature of the yam is 120C.*

(a)

$$\begin{aligned} \frac{dH}{dt} &= -k(H - 200) \\ \text{Separate variables:} \quad \frac{dH}{H - 200} &= -kdt \\ \text{Integrating both sides:} \quad \ln|H - 200| &= -kt + C \\ \text{Exponentiate both sides:} \quad e^{\ln|H-200|} = |H - 200| &= e^{-kt+C} \\ H - 200 &= Ae^{-kt} \quad \text{if we let } A = \pm e^C \\ H &= 200 + Ae^{-kt} \\ \text{If } H(0) = 20, \quad 20 &= 200 + Ae^0 \\ A &= -180 \\ \text{so} \quad H &= 200 - 180e^{-kt} \end{aligned}$$

The temperature of the yam over time is given by the formula $H = 200 - 180e^{-kt}$.

(b)

$$\begin{aligned}
\text{If} \quad H &= 200 - 180e^{-kt} \\
\text{and} \quad H(30) &= 120 \\
\text{then} \quad 120 &= 200 - 180e^{-30k} \\
\text{Solve for } k: \quad e^{-30k} &= \frac{-80}{-180} = \frac{4}{9} \\
k &= \frac{1}{30}\ln\left(\frac{4}{9}\right) \approx 0.027.
\end{aligned}$$

Note that this constant is correct if we measured t in minutes. The calculation would be identical except for a factor of 60 if you measured t in hours.

7. *The rate of growth of a tumor is proportional to the size of the tumor.*

   (a) *Write a differential equation satisfied by S, the size of the tumor, in mm, as a function of time, t.*
   (b) *Find the general solution to the differential equation.*
   (c) *If the tumor is 5 mm across at time $t = 0$, what does that tell you about the solution?*
   (d) *If, in addition, the tumor is 8 mm across at time $t = 3$, what does that tell you about the solution?*

   (a) $\dfrac{dS}{dt} = kS.$
   (b) You can solve this by separation of variables, or by recognizing that the solution is exponential.
   $S = Ae^{kt}$
   (c) If $S(0) = 5$, that allows us to find the value of A:

   $$\begin{aligned} 5 &= Ae^{k\cdot 0} \\ A &= 5 \end{aligned}$$

   (d) If $S(3) = 8$, that allows us to find k:

   $$\begin{aligned} 8 &= 5e^{3k} \\ k &= \frac{1}{3}\ln\left(\frac{8}{5}\right) \approx 0.1567 \end{aligned}$$

20. *The amount of land in use for growing crops increases as the world's population increases. Suppose $A(t)$ represents the total number of hectares of land in use in year t. (A hectare is about $2\frac{1}{2}$ acres.)*

   (a) *Explain why it is plausible that $A(t)$ satisfies the equation $A'(t) = kA(t)$. What assumptions are you making about the world's population and its relation to the amount of land used?*

(b) *In 1950 about* $1 \cdot 10^9$ *hectares of land were in use; in 1980 the figure was* $2 \cdot 10^9$. *If the total amount of land available for growing crops is thought to be* $3.2 \cdot 10^9$ *hectares, when does this model predict it is exhausted? (Let* $t = 0$ *in 1950.)*

(a) If the world population is growing roughly exponentially, and it requires a fixed amount of arable land to support each person, then it makes sense that the rate of arable land must also be increasing roughly exponentially. An exponential rate of increase can be specified by the DE $A'(t) = kA(t)$.

Obviously, this is a crude approximation. First, you would need to demonstrate that the world population is in fat growing exponentially. Secondly, you would have to discount the effect of technology in both making more land productive for agriculture, and its effect on increasing (or decreasing) crop yields per hectare.

(b) The solution to the DE is $A(t) = A_0 e^{kt}$.

$$\begin{aligned} A(0) = A_0 &= 10^9 && \text{if } t = 0 \text{ represents 1950} \\ A(30) = 2 \times 10^9 &= 10^9 e^{30k} && \text{as } t = 30 \text{ represents 1980} \\ 2 &= e^{30k} \\ k &= \frac{1}{30} \ln 2 \approx 0.0231 \end{aligned}$$

Solve for t when $A(t) = 3.2 \times 10^9$

$$\begin{aligned} 3.2 \times 10^9 &= 10^9 e^{0.0231t} \\ t &= \frac{\ln(3.2)}{0.0231} \approx 50 \text{ years} \end{aligned}$$

According to this model, we will have every arable hectare in use at $t = 50$, or in the year 2000. Hmmmm.

21. *A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3F. One hour later, the temperature of the body is 89.0F. The temperature of the room has been maintained at a constant 68F.*

(a) *Assuming the temperature,* T, *of the body obeys Newton's Law of Cooling, write a differential equation for* T.

(b) *Solve the differential equation to estimate the time the murder occurred.*

(a) Let T represent the temperature of the body. Newton's Law of Heating and Cooling states that

$$\frac{dT}{dt} = -k(T - T_{room}) = -k(T - 68)$$

(b) Solve by separation of variables:

$$\begin{aligned}
& & \frac{dT}{T-68} &= -k\ dt \\
&\text{Integrating:} & \ln|T-68| &= -kt + C \\
&\text{Exponentiate:} & T-68 &= Ae^{-kt} \quad \text{if } A = \pm e^C \\
& & T &= Ae^{-kt} + 68 \\
&\text{Let 9 AM be } t=0: & 90.3 &= A + 68 \\
& & A &= 22.3 \\
&\text{Let 10 AM be } t=1: & 89.0 &= 22.3e^{-k} + 68 \\
& & k &= 0.06006
\end{aligned}$$

Solve for time when T was 98.6 = normal body temp:

$$\begin{aligned}
98.6 &= 22.3e^{-0.06006t} + 68 \\
t &= -5.27
\end{aligned}$$

It looks like the time of death was roughly 5 hours before 9 AM, or just before 4 AM.

24. *Before Galileo discovered that the speed of a falling body with no air resistance is proportional to the time since it was dropped, he mistakenly conjectured that the speed was proportional to the distance it had fallen.*

   (a) *Assume the mistaken conjecture to be true and write an equation relating the distance fallen, $D(t)$, at time t, and its derivative.*

   (b) *Using your answer to part (a) and the correct initial conditions, show that D would have to be equal to 0 for all t, and therefore the conjecture must be wrong.*

(a)

$$\begin{aligned}
\text{Velocity } &\propto \text{Distance} \\
\frac{dD}{dt} &\propto D(t) \\
\frac{dD}{dt} &= kD
\end{aligned}$$

(b) If we started at position $D(0) = 0$, this would give a derivative of $\frac{dD}{dt} = k(0) = 0$. A zero derivative indicates that the function itself is constant, so D stays at zero. Since $D = 0$ will lead to $\frac{dD}{dt} = 0$ for any value of t, we get $D(t) = 0$.
In other words, when we drop an object, and has not yet moved, it will not accelerate. Hmmm....