SUGGESTED PROBLEMS
In Exercises 15-28 find the gradient of the given function. Assume the variables are restricted to a domain on which the function is defined.

15. \( f(x, y) = \frac{3}{2}x^5 - \frac{4}{7}y^6 \)

\[
\begin{align*}
  f_x &= \frac{15}{2}x^4 \\
  f_y &= -\frac{24}{7}y^5
\end{align*}
\]

so \( \nabla f = \left( \frac{15}{2}x^4 \right) \hat{i} + \left( -\frac{24}{7}y^5 \right) \hat{j} \)

or more simply, in component notation,

\[
= \left( \frac{15}{2}x^4, -\frac{24}{7}y^5 \right)
\]

21. \( f(x, y) = \sqrt{x^2 + y^2} \)

\( f(x, y) = (x^2 + y^2)^{1/2} \)

\[
\begin{align*}
  f_x &= \frac{1}{2}(x^2 + y^2)^{-1/2} (2x) = \frac{x}{\sqrt{x^2 + y^2}} \\
  f_y &= \frac{y}{\sqrt{x^2 + y^2}}
\end{align*}
\]

similarly, \( f_y = \frac{y}{\sqrt{x^2 + y^2}} \)

so \( \nabla f = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \)

31. \( f(r, h) = 2\pi rh + \pi r^2, \) at \((2, 3)\)

\[
\begin{align*}
  f_r &= 2\pi h + 2\pi r = 2\pi (r + h) \\
  f_h &= 2\pi r \\
  \nabla f(2, 3) &= (2\pi (2 + 3), 2\pi (2)) \\
  &= (10\pi, 4\pi)
\end{align*}
\]
33. \( f(x, y) = 1/(x^2 + y^2) \), at \((-1, 3)\)

It will help to write \( f(x, y) = (x^2 + y^2)^{-1} \).

\[
\begin{align*}
  f_x &= -1(x^2 + y^2)^{-2}(2x) \\
  &= \frac{-2x}{(x^2 + y^2)^2} \\
  f_y &= \frac{-2y}{(x^2 + y^2)^2} \\
  \text{so} \quad \nabla f(-1, 3) &= \left\langle \frac{-2(-1)}{(1 + 9)^2}, \frac{-2(3)}{(1 + 9)^2} \right\rangle \\
  &= \left\langle \frac{2}{100}, \frac{-6}{100} \right\rangle
\end{align*}
\]

**QUIZ PREPARATION PROBLEMS**

*In Exercises 1-6, use the contour diagram of \( f \) in Figure 14.31 to decide if the specified directional derivative is positive, negative, or approximately zero.*

1. **At point \((-2, 2)\), in direction \( \vec{i} \).**

   Moving from contour \( z = 6 \) towards contour \( z = 4 \) means \( z \) is decreasing in that direction, so the directional derivative is negative.

2. **At point \((0, -2)\), in direction \( \vec{j} \).**

   Moving from \( z = 4 \) towards \( z = 2 \), so directional derivative is negative.

3. **At point \((-1, 1)\), in direction \( \vec{i} + \vec{j} \)**

   We are moving parallel to the contour at that point, so our \( z \) value would be unchanging at that instant, so the directional derivative is zero.
4. At point \((-1, 1)\), in direction \(-\mathbf{i} + \mathbf{j}\).
   Moving towards higher \(z\) values, so the directional derivative is positive.

5. At point \((0, -2)\), in direction \(\mathbf{i} + 2\mathbf{j}\).
   Moving more in the \(y\) direction than \(x\), or towards lower \(z\) values, so the derivative is negative.

6. At point \((0, -2)\), in direction \(\mathbf{i} - 2\mathbf{j}\).
   Moving towards higher \(z\) values, so the derivative is positive.

25. \(z = \sin(x/y)\)

\[
\frac{\partial z}{\partial x} = \cos \left( \frac{x}{y} \right) \frac{1}{y} \\
\frac{\partial z}{\partial y} = \cos \left( \frac{x}{y} \right) \frac{-x}{y^2}
\]

so \(\nabla f = \left\langle \frac{1}{y} \cos \left( \frac{x}{y} \right), \frac{-x}{y^2} \cos \left( \frac{x}{y} \right) \right\rangle\)

27. \(f(\alpha, \beta) = \frac{2\alpha + 3\beta}{2\alpha - 3\beta}\)

\[
f_\alpha = \frac{2(2\alpha - 3\beta) - (2\alpha + 3\beta)(2)}{(2\alpha - 3\beta)^2} = \frac{-12\beta}{(2\alpha - 3\beta)^2}
\]
\[
f_\beta = \frac{3(2\alpha - 3\beta) - (2\alpha + 3\beta)(-3)}{(2\alpha - 3\beta)^2} = \frac{12\alpha}{(2\alpha - 3\beta)^2}
\]

so \(\nabla f = \left\langle \frac{-12\beta}{(2\alpha - 3\beta)^2}, \frac{12\alpha}{(2\alpha - 3\beta)^2} \right\rangle\)

In Exercises 29-34, find the gradient at the given point.

29. \(f(x, y) = x^2 y + 7xy^3\), at \((1, 2)\)

\[
f_x = 2xy + 7y^3 \\
f_y = x^2 + 21xy^2
\]

so at \((1, 2)\), \(\nabla f(1, 2) = \langle 2(1)(1) + 7(2)^3, 1^2 + 21(1)(2)^2 \rangle = \langle 60, 85 \rangle\)

In Exercises 35-38, find the directional derivative \(f_\vec{u}(1, 2)\) for the function \(f\) with \(\vec{u} = (3\mathbf{i} - 4\mathbf{j})/5\).
35. \( f(x, y) = 3x - 4y \)

We note that the vector \( \vec{u} \) is already a unit vector, since \( ||\vec{u}|| = \sqrt{\frac{3^2}{5} + \frac{4^2}{5^2}} = \sqrt{\frac{25}{25}} = 1. \)

This means that we can compute the directional derivative using the simple formula

\( f_{\vec{u}}(1, 2) = (\nabla f(1, 2)) \cdot \vec{u} \).

We will use the component for of \( \vec{u} = \frac{1}{5}(3, -4) \)

\[
\begin{align*}
  f_x &= 3 \\
  f_y &= -4 \\
  \text{so} \quad \nabla f(1, 2) &= (3, -4) \\
  \text{So} \quad f_{\vec{u}}(1, 2) &= (\nabla f(1, 2)) \cdot \vec{u} \\
  &= (3, -4) \cdot \left\langle \frac{1}{5}(3, -4) \right\rangle \\
  &= \frac{1}{5}(9 + 16) \\
  &= 5
\end{align*}
\]

37. \( f(x, y) = xy + y^3 \)

\[
\begin{align*}
  f_x &= y \\
  f_y &= x + 3y^2 \\
  \text{so} \quad \nabla f(1, 2) &= (2, 1 + 3(2^2)) = (2, 13) \\
  \text{So} \quad f_{\vec{u}}(1, 2) &= (2, 13) \cdot \left\langle \frac{1}{5}(3, -4) \right\rangle \\
  &= \frac{1}{5}(6 - 52) = \frac{-46}{5}
\end{align*}
\]

66. The temperature at any point in the plane is given by the function

\[
T(x, y) = \frac{100}{x^2 + y^2 + 1}
\]

(a) What shape are the level curves of \( T \)?

(b) Where on the plane is it hottest? What is the temperature at that point?

(c) Find the direction of the greatest increase in temperature at the point \( (3, 2) \). What is the magnitude of that greatest increase?

(d) Find the direction of the greatest decrease in temperature at the point \( (3, 2) \).

(e) Find a direction at the point \( (3, 2) \) in which the temperature does not increase or decrease.
(a) The level curves of $T$ are circles, since if $T$ is a constant values, say $c$:

\[
c = \frac{100}{x^2 + y^2 + 1}
\]

\[
x^2 + y^2 + 1 = \frac{100}{c}
\]

so \[
x^2 + y^2 = \frac{100}{c} - 1
\]

which is the formula for a circle in 2D.

(b) The plane will be hottest when the denominator is smallest. From the form of the denominator, this will be when $x^2$ and $y^2$ are smallest, or at $(x, y) = (0, 0)$. Any other point $(x, y)$ leads to a smaller temperature.

(c) To find the direction of maximum temperature increase at $(3, 2)$, we need the gradient vector.

\[
T_x = \frac{-200x}{(x^2 + y^2 + 1)^2}
\]

\[
T_y = \frac{-200y}{(x^2 + y^2 + 1)^2}
\]

\[
\nabla T(3, 2) = \left\langle \frac{-600}{196}, \frac{-400}{196} \right\rangle
\]

or, to see the direction more easily,

\[
\nabla T(3, 2) = \frac{200}{196} \left\langle -3, -2 \right\rangle = \frac{50}{49} \left\langle -3, -2 \right\rangle
\]

The steepness of the slope in this direction is given by $||\nabla T||$:

\[
||\nabla T(3, 2)|| = \frac{50}{49} ||\left\langle -3, -2 \right\rangle|| = \frac{50}{49} \sqrt{13} \approx 3.68
\]

At $(3, 2)$, if you move towards the origin (in the direction of $\langle -3, -2 \rangle$), the temperature will increase at a rate of 3.68 degrees per unit distance.

(d) If the direction of maximum temperature increase is $\langle -3, -2 \rangle$, then going in the opposite direction, $\langle 3, 2 \rangle$, will produce the most rapid temperature decrease.

(e) The direction in which the temperature does not change is perpendicular to gradient, or in the direction $\langle 2, -3 \rangle$ or $\langle -2, 3 \rangle$. 