Unit #5 - Implicit Differentiation, Related Rates

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Implicit Differentiation

1. Consider the graph implied by the equation \( xy^2 = 1 \).

What is the equation of the line through \((\frac{1}{2}, 2)\) which is also tangent to the graph?

Differentiating both sides with respect to \( x \),

\[
y^2 + x \left( 2y \frac{dy}{dx} \right) = 0
\]

so

\[
\frac{dy}{dx} = -\frac{y^2}{2xy}
\]

At the point \((\frac{1}{2}, 2)\), \( \frac{dy}{dx} = -4 \), so we can use the point/slope formula to obtain the tangent line

\[
y = -4(x - \frac{1}{2}) + 2
\]

2. Consider the circle defined by \( x^2 + y^2 = 25 \)

(a) Find the equations of the tangent lines to the circle where \( x = 4 \).

(b) Find the equations of the normal lines to this circle at the same points. (The normal line is perpendicular to the tangent line.)

(c) At what point do the two normal lines intersect?

Differentiating both sides,

\[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} \quad (25)
\]

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}
\]

(a) The points at \( x = 4 \), satisfying \( x^2 + y^2 = 25 \), would be \( y = 3 \) and \( y = -3 \).

Using the point/slope formula for a line, and our calculated \( \frac{dy}{dx} \) through (4, 3): \( y = -1.33333 \cdot (x - 4) + 3 \), and through (4, -3): \( y = 1.33333 \cdot (x - 4) + (-3) \)

(b) If you have a line with slope \( m \), the slope of a perpendicular line will be \(-1/m\).

Through (4, 3): \( y = \frac{3}{4}(x - 4) + 3 \), and through (4, -3): \( y = \frac{3}{4}(x - 4) + (-3) \)

(c) Setting the \( y \) value in both equations equal to each other and solving to find the intersection point, the only solution is \( x = 0 \) and \( y = 0 \), so the two normal lines will intersect at the origin, \((x, y) = (0, 0)\).

Solving process:

\[
\begin{align*}
\frac{3}{4}(x - 4) + 3 &= -\frac{3}{4}(x - 4) - 3 \\
\frac{3}{4}x - 3 + 3 &= -\frac{3}{4}x - 3 \\
\frac{3}{4}x &= -\frac{3}{4} \\
6 &= 0 \\
x &= 0
\end{align*}
\]

Subbing back into either equation to find \( y \), e.g. \( y = \frac{3}{4}(0 - 4) + 3 = -3 + 3 = 0 \).

3. Calculate the derivative of \( y \) with respect to \( x \), given that

\[
x^4y + 4xy^4 = x + y
\]

Let \( x^4y + 4xy^4 = x + y \). Then

\[
4x^3y + x^4 \frac{dy}{dx} + (4x) \left( 4y^3 \frac{dy}{dx} \right) + 4y^4 = 1 + \frac{dy}{dx}
\]

hence

\[
\frac{dy}{dx} = \frac{4yx^3 + 4y^4 - 1}{1 - x^4 - 16xy^3}
\]

4. Calculate the derivative of \( y \) with respect to \( x \), given that

\[
x e^{xy} = 4xy + 5y^4
\]

To solve for \( \frac{dy}{dx} \), we must think of \( y \) as a function of \( x \) and differentiate both sides of the equation, using the chain rule where appropriate:

\[
e^y + x e^{xy} \frac{dy}{dx} = 4y + 4x \frac{dy}{dx} + 20y^3 \frac{dy}{dx}
\]

Now, we simplify and move the terms with a \( \frac{dy}{dx} \) to the right, and keep the terms without a \( \frac{dy}{dx} \) to the left:

\[
e^y - 4y = (4x + 20y^3 - xe^y) \frac{dy}{dx}
\]

Finally, we can solve for \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \frac{e^y - 4y}{4x + 20y^3 - xe^y}
\]
5. Use implicit differentiation to find the equation of the tangent line to the curve \( xy^2 + xy = 14 \) at the point \( (7, 1) \).

To get the slope, we take the derivative \( \frac{d}{dx} \) of both sides.

\[
\frac{d}{dx} (xy^2 + xy) = \frac{d}{dx} (14)
\]

\[
[1)y^3 + x \left(3y \frac{dy}{dx}\right)] + [(1)y + x \left(\frac{dy}{dx}\right)] = 0
\]

Gathering terms with \( \frac{dy}{dx} \), and those without it,

\[
\frac{dy}{dx} (3xy^2 + x) + (y^3 + y) = 0
\]

\[
\frac{dy}{dx} = -\frac{y^3 + y}{3xy^2 + x}
\]

At the point \( (x, y) = (7, 1) \),

\[
\frac{dy}{dx} = -\frac{1^3 + 1}{3(7)(1)^2 + 7}
\]

\[
= -\frac{2}{28} = -\frac{1}{14}
\]

To build a line that goes through the point \( (7, 1) \), and with slope \( -\frac{1}{14} \), we can use the point-slope line formula. We obtain the formula for the tangent line:

\[
y = -\frac{1}{14} (x - 7) + 1
\]

6. Find \( \frac{dy}{dx} \) by implicit differentiation.

\[\sqrt{x+y} = 9 + x^2y^2\]

Taking \( \frac{d}{dx} \) of both sides,

\[
\frac{d}{dx} \left( \sqrt{x+y} \right) = \frac{d}{dx} \left( 9 + x^2y^2 \right)
\]

\[
\frac{1}{2} \frac{1}{\sqrt{x+y}} \left( 1 + \frac{dy}{dx} \right) = 2xy^2 + x^22y \left( \frac{dy}{dx} \right)
\]

Expanding left side:

\[
\left( \frac{1}{2} \frac{1}{\sqrt{x+y}} \right) + \frac{dy}{dx} \left( \frac{1}{2} \frac{1}{\sqrt{x+y}} \right) = 2xy^2 + x^22y \left( \frac{dy}{dx} \right)
\]

Gathering terms with and without \( \frac{dy}{dx} \),

\[
\frac{dy}{dx} \left( \frac{1}{2\sqrt{x+y}} - 2x^2y \right) = 2xy^2 - \frac{1}{2\sqrt{x+y}}
\]

To simplify a little we multiply both sides through by \( 2\sqrt{x+y} \):

\[
\frac{dy}{dx} (1 - 4x^2y\sqrt{x+y}) = 4xy^2 \sqrt{x+y} - 1
\]

\[
\frac{dy}{dx} = \frac{(4xy^2 \sqrt{x+y} - 1)}{1 - 4x^2y\sqrt{x+y}}
\]

7. Find all the x-coordinates of the points on the curve \( x^2y^2 + xy = 2 \) where the slope of the tangent line is \(-1\).

We need to find the derivative \( \frac{dy}{dx} \) by implicit differentiation. Differentiating with respect to \( x \) on both sides of the equation,

\[
2xy^2 + x^2 \left( 2y \frac{dy}{dx} \right) + y + x \frac{dy}{dx} = 0
\]

Here, we could solve for \( \frac{dy}{dx} \), but we actually know the slope we want this time: \( \frac{dy}{dx} = -1 \), so let’s just sub that in now:

\[
2xy^2 + x^2 (2y) + y + x (-1) = 0
\]

or \( 2xy^2 - 2x^2 y - y - x = 0 \)

factoring first terms: \( 2xy(y-x) + (y-x) = 0 \)

Factoring common \( y-x \): \( (y-x)(2xy + 1) = 0 \)

Meaning either \( y-x = 0 \) (so \( y = x \)), or \( 2xy+1 = 0 \).

Subbing each possibility into the equation for the curve, \( x^2y^2 + xy = 2 \)

If \( y = x \), \( x^2(x^2) + x(x) = 2 \)

\[
x^4 + x^2 = 2
\]

\[
x^4 + x^2 - 2 = 0
\]

\[
(x^2 + 2)(x^2 - 1) = 0
\]

So \( x = -1, 1 \) are two solutions, with the corresponding \( y \) values, using \( y = x \) being \( y = -1 \) and 1 as well.

The other possible case, \( 2xy + 1 = 0 \), leads to

If \( y = -1/2x \), \( x^2 \left( \frac{-1}{2x} \right)^2 + x \left( \frac{-1}{2x} \right) = 2 \)

or \( \frac{1}{4} + \frac{-1}{2} = 2 \)

which is impossible, so the assumption that \( y = -1/2x \) must be impossible to use with this curve.

Therefore the only two points on the curve \( x^2y^2 + xy = 2 \) which have slope \( \frac{dy}{dx} = -1 \) are \( (x, y) = (1, 1) \) and \( (-1, -1) \).
8. Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse for the second time?

To obtain a normal (perpendicular) line, we find a line perpendicular to the tangent line on the ellipse at the point $(-1, 1)$. (Linear algebra students may have other ways to do this.)

To get the slope of the tangent line, we use implicit differentiation, with respect to $x$:

$$\frac{d}{dx} (x^2 - xy + y^2) = \frac{d}{dx} (3)$$

$$2x - (y + x \frac{dy}{dx}) + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (-x + 2y) = -2x + y$$

$$\frac{dy}{dx} = \frac{-2x + y}{(-x + 2y)}$$

At the point on the ellipse $(-1, 1)$,

$$\frac{dy}{dx} = \frac{-2(-1) + 1}{(-(-1) + 2(1))} = \frac{3}{3} = 1$$

So the slope of the tangent line to the ellipse at $(-1, 1)$ is $1$.

The slope of the normal line will be perpendicular to that, or $-1/(1) = -1$.

The normal line, which also passes through $(-1, 1)$, will therefore be

$$y = -1(x - (-1)) + 1$$

or $y = -1(x + 1) + 1$

or $y = -x$

To find the intersections of this normal line with the ellipse to find a second crossing, we sub in this expression for $y$ into the ellipse formula:

$$x^2 - xy + y^2 = 3$$

$$x^2 - x(-x) + (-x)^2 = 3$$

$$x^2 + x^2 + x^2 = 3$$

$$x^2 = 1x = \pm 1$$

Since $x = -1$ is the point we started at, $x = +1$ must be the other intersection. At that point, $y = -x$, so the point is $(x, y) = (1, -1)$. Below is a graph of the scenario.

9. The curve with equation $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$ has been likened to a bouncing wagon (graph it to see why). Find the $x$-coordinates of the points on this curve that have horizontal tangents.

Taking an implicit $\frac{dx}{dy}$ of both sides,

$$\frac{d}{dy} \left( 2y^3 + y^2 - y^5 \right) = \frac{d}{dx} \left( x^4 - 2x^3 + x^2 \right)$$

$$6y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} - 5y^4 \frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

Since we know we want points where $\frac{dy}{dx} = 0$ (horizontal tangents), we’ll set $\frac{dy}{dx} = 0$ immediately:

$$6y^2(0) + 2y(0) - 5y^4(0) = 4x^3 - 6x^2 + 2x$$

factoring: $0 = 2x(2x^2 - 3x + 1)$

$0 = 2x(2x - 1)(x - 1)$

So $x = 0, \frac{1}{2}$ and $1$ are the points where the tangent to the graph will be horizontal.

Note that the question only asked for the $x$ coordinate of these points. As you can see on the graph below, all of these $x$ coordinates correspond to multiple $y$ values, but all of those points have horizontal tangents.
10. Use implicit differentiation to find an equation of the tangent line to the curve

\[ y^2(y^2 - 4) = x^2(x^2 - 5) \]

at the point \((x, y) = (0, -2)\).

This point happens to be at the bottom of the loop on the \(y\) axis, so the slope there is zero. Therefore, the tangent line equation is simply:

\[ y = -2 \]

11. Use implicit differentiation to find the \((x, y)\) points where the circle defined by

\[ x^2 + y^2 - 2x - 4y = -1 \]

has horizontal and vertical tangent lines.

(a) Find the points where the curve has a horizontal tangent line.

(b) Find the points where the curve has a vertical tangent line.

(a) Use implicit differentiation, then set \( \frac{dy}{dx} = 0 \).

\[
\frac{d}{dx} (x^2 + y^2 - 2x - 4y) = \frac{d}{dx} (-1) \\
2x + 2y \frac{dy}{dx} - 2 - 4 \frac{dy}{dx} = 0 \\
\frac{dy}{dx} (2y - 4) = -2x + 2 \\
\frac{dy}{dx} = \frac{-2x + 2}{2y - 4}
\]

Setting \( \frac{dy}{dx} \) now equal to zero gives

\[
0 = \frac{-2x + 2}{2y - 4} \\
-2x + 2 = 0 \\
x = 1
\]

Substituting \( x = 1 \) back into the circle equation to find the matching \( y \) coordinates gives

\[
(1)^2 + y^2 - 2(1) - 4y = -1 \\
y^2 - 4y = 0 \\
y(y - 4) = 0 \\
y = 0, 4
\]

The points with horizontal tangents are \((1,0)\) and \((1,4)\).

(b) The points with vertical tangents are those where the denominator of \( \frac{dy}{dx} \) is zero (making the slope undefined). From part (a), we have

\[
\frac{dy}{dx} = \frac{-2x + 2}{2y - 4}
\]

Setting the denominator equal to zero gives

\[
2y - 4 = 0 \\
y = 2
\]

Substituting \( y = 2 \) back into the circle equation to find the matching \( x \) coordinates gives

\[
x^2 + (2)^2 - 2x - 4(2) = -1 \\
x^2 - 2x - 3 = 0 \\
(x - 3)(x + 1) = 0 \\
x = 3, -1
\]

This gives points on the circle with vertical tangents at \((3,2)\), and \((-1,2)\).

12. The relation

\[ x^2 - 2xy + y^2 + 6x - 10y + 29 = 0 \]

defines a parabola.

(a) Find the points where the curve has a horizontal tangent line.

(b) Find the points where the curve has a vertical tangent line.

(a) Use implicit differentiation, then set \( \frac{dy}{dx} = 0 \).

The only point with horizontal tangent is \((2, 5)\).

(b) From the implicit derivative, get the equation in the form \( \frac{dy}{dx} = \ldots \).

The point with vertical tangent is the point where the denominator of \( \frac{dy}{dx} \) is zero, or \((1, 6)\).
13. The graph of the equation
\[ x^2 + xy + y^2 = 9 \]
is a slanted ellipse illustrated in this figure:

Think of \( y \) as a function of \( x \).

(a) Differentiating implicitly, find a formula for the slopes of this shape. (Your answer will depend on \( x \) and \( y \).)

(b) The ellipse has two horizontal tangents. The upper one has the equation characterized by \( y' = 0 \). To find the vertical tangent use symmetry, or think of \( x \) as a function of \( y \), differentiate implicitly, solve for \( x' \) and then set \( x' = 0 \).

Taking \( \frac{d}{dx} \) of both sides of the relation
\[ x^2 + xy + y^2 = 9 \]
gives
\[ 2x + y + xy' + 2yy' = 0. \]

Solving for \( y' \) gives
\[ y' = -\frac{2x + y}{x + 2y}. \]

Setting
\[ y' = 0 \]
and solving for \( x \) gives
\[ 2x + y = 0 \quad \implies \quad x = -\frac{y}{2}. \]

Substituting in the original equation gives
\[ \frac{y^2}{4} - \frac{y^2}{2} + y^2 = 9. \]
Thus
\[ \frac{3y^2}{4} = 9 \]
or
\[ y^2 = 12. \]

Hence the horizontal tangent lines have the equations
\[ y = \pm \sqrt{12}. \]

The plus sign gives the upper tangent line, the minus sign the lower. By symmetry the vertical tangent lines have the equations
\[ x = \pm \sqrt{12}. \]

The rightmost vertical tangent line with the equation
\[ x = 6 \]
touches the ellipse in the point where
\[ y = \frac{x}{2} = -\frac{\sqrt{12}}{2} = -\sqrt{3}. \]

---

Related Rates

14. Gravel is being dumped from a conveyor belt at a rate of 30 cubic feet per minute. It forms a pile in the shape of a right circular cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 17 feet high?

Recall that the volume of a right circular cone with height \( h \) and radius of the base \( r \) is given by \( V = \frac{1}{3} \pi r^2 h \).

Let \( V(t) \) = volume of the conical pile, and \( h \) and \( r \) be the height and bottom radius of the cone respectively. The cone has a base diameter and \( h \) that are equal, so \( r = h/2 \). We have \( \frac{dV}{dt} \), and want \( \frac{dh}{dt} \).

Always true:
\[ V = \frac{1}{3} \pi r^2 h \]
but since \( r = h/2 \) for this cone,
\[ V = \frac{\pi h^2}{3} \cdot \frac{h}{4} \]

Taking \( \frac{d}{dt} \) of both sides:
\[ \frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt} \]
so
\[ \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt} \]

Subbing in \( \frac{dV}{dt} = 30 \text{ ft}^3/\text{min} \), and \( h = 17 \),
15. When air expands adiabatically (without gaining or losing heat), its pressure $P$ and volume $V$ are related by the equation $PV^{1.4} = C$ where $C$ is a constant. Suppose that at a certain instant the volume is 550 cubic centimeters, and the pressure is 91 kPa and is decreasing at a rate of 7 kPa/minute. At what rate in cubic centimeters per minute is the volume increasing at this instant?

(Pa stands for Pascal – it is equivalent to one Newton/(meter squared); kPa is a kiloPascal or 1000 Pascals.)

Let $P$ and $V$ be the pressure and volume respectively, and $C$ be a constant. We know $\frac{dP}{dt}$, and want $\frac{dV}{dt}$.

Always true:

$PV^{1.4} = C$

Taking $\frac{d}{dt}$:

$\frac{d}{dt} (PV^{1.4}) = \frac{d}{dt} (C)$

$\frac{dP}{dt} V^{1.4} + P \left(1.4V^{0.4}\frac{dV}{dt}\right) = 0$

$\frac{dP}{dt} V^{1.4} + P \left(1.4V^{0.4}\frac{dV}{dt}\right) = 0$

so

$\frac{dV}{dt} = -\frac{\frac{dP}{dt} V^{1.4}}{1.4PV^{0.4}}$

Subbing in the known values gives

$\frac{dV}{dt} = 30.22 \text{ cm}^3/\text{min}$

16. At noon, ship A is 40 nautical miles due west of ship B. Ship A is sailing west at 23 knots and ship B is sailing north at 23 knots. How fast (in knots) is the distance between the ships changing at 6 PM? (Note: 1 knot is a speed of 1 nautical mile per hour.)

The two boats can always be drawn in a triangular configuration. Let their east/west (horizontal on the graph) distance be $x$, and their north/south distance be $y$.

From this, the distance between the boats is $D$. We know $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and want $\frac{dD}{dt}$.

Always true:

$D^2 = x^2 + y^2$

Taking $\frac{d}{dt}$:

$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$

$\frac{dD}{dt} = \frac{1}{D} \left( x \frac{dx}{dt} + y \frac{dy}{dt}\right)$

At noon, ship A is due west of ship B, so $x = 40$ and $y = 0$ nautical miles.

At the question time of 6 PM, each ship will have traveled $23 \times 6 = 138$ nautical miles, so $x = 40 + 138 = 178$ and $y = 138$. This gives $D = \sqrt{178^2 + 138^2} \approx 225.23$.

Also $\frac{dx}{dt} = 23$ and $\frac{dy}{dt} = 23$ knots (nautical miles per hour). Subbing in these values into the derivative formula above gives

$\frac{dD}{dt} = 32.269$ knots

17. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 3 cm²/min.

At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 105 cm²?

Let $x$ be the base of the triangle, and $y$ be the height, so area $A = \frac{1}{2}xy$.

(Repeat that the triangle doesn’t have to be a right-angled triangle as shown, since only the base and height affect the area, not the internal angles.)

We are given $\frac{dy}{dt}$ and $\frac{dA}{dt}$, and want $\frac{dx}{dt}$.

Always true:

$A = \frac{1}{2}xy$

Taking $\frac{d}{dt}$:

$\frac{dA}{dt} = \frac{1}{2} \left( \frac{dx}{dt} y + x \frac{dy}{dt}\right)$

Solving for $\frac{dx}{dt}$:

$\frac{dx}{dt} = \frac{1}{y} \left( 2 \frac{dA}{dt} - x \frac{dy}{dt} \right)$

At the point where $y = 10$ cm and $A = 105$, we can find $x = 2A/y = 21$ cm. We are given $\frac{dx}{dt} = 1$ cm/min and $\frac{dA}{dt} = 3$ cm/min. Subbing these values into $\frac{dx}{dt}$,

$\frac{dx}{dt} = -1.5$ cm/min
18. A street light is at the top of a 11 ft tall pole. A woman 6 ft tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the tip of her shadow moving when she is 45 ft from the base of the pole?

The tip of the shadow is moving at 11 ft/s (rate of change of the distance from the shadow tip to the fixed point of the pole).

A related quantity, the size of the shadow (distance from the woman to the end of the shadow), or \( \frac{ds}{dt} \) in the diagram, is growing at 6 ft/s.

The question asks for the speed of the tip so 11 ft/s is the correct answer here.

19. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat.

If the rope is pulled in at a rate of 1.2 m/s, how fast is the boat approaching the dock when it is 9 m from the dock?

We define variables for the two lengths that are changing:

- \( x \) = length of the rope, and
- \( w \) = water-level distance between the boat and the dock.

In the problem, we are told that the rope is being pulled in at 1.2 m/s, so \( \frac{dx}{dt} = -1.2 \). We want to find the speed of the boat relative to the water, or \( \frac{dw}{dt} \).

To get the relationship between those two rates, we start with an equation that is always true:

\[
w^2 + 1^2 = x^2
\]

Differentiating both sides with respect to time,

\[
\frac{d}{dt}(w^2 + 1) = \frac{d}{dt}x^2
\]

\[
2w \frac{dw}{dt} = 2x \frac{dx}{dt}
\]

Note that we have \( \frac{dx}{dt} \), but we still need \( x \) and \( w \).
We are asked for the water speed when \( w = 9 \), but we also need \( x \) at that point. Using the pythagorean equation again,

\[ 9^2 + 1 = x^2 \]
\[ x = \sqrt{82} \]

Subbing in to our related rates equation,

\[ 2w \frac{dw}{dt} = 2x \frac{dx}{dt} \]
\[ 2(9) \frac{dw}{dt} = 2(\sqrt{82})(-1.2) \]
\[ \frac{dw}{dt} = \frac{(\sqrt{82})}{9}(-1.2) \]
\[ = -1.207 \text{ m/s} \]

The boat is approaching the dock (distance is decreasing) at 1.207 m/s.

20. A plane flying with a constant speed of 4 km/min passes over a ground radar station at an altitude of 16 km and climbs at an angle of 45 degrees. At what rate, in km/min, is the distance from the plane to the radar station increasing 1 minute later?

Recall the cosine law,

\[ c^2 = a^2 + b^2 - 2ab \cos \left( \frac{3\pi}{4} \right) \]

where \( \theta \) is the angle between the sides of length \( a \) and \( b \).

Here is a sketch of the scenario. Note that both \( b \) and \( c \) are increasing as the plane flies away.

The cosine law gives us the “always true” relationship, based on the constant angle of \( \frac{3\pi}{4} \):

\[ c^2 = a^2 + b^2 - 2ab \cos \left( \frac{3\pi}{4} \right) \]

Filling in \( a = 16 \) and \( \cos \left( \frac{3\pi}{4} \right) = -\frac{1}{\sqrt{2}} \),

\[ c^2 = 16^2 + b^2 - 2(16)b \left( -\frac{1}{\sqrt{2}} \right) \]
\[ c^2 = 256 + b^2 + \frac{32}{\sqrt{2}}b \]

We know \( \frac{db}{dt} = 4 \text{ km/min} \) (plane’s speed), and we want to know \( \frac{dc}{dt} \).

To get the relationship between those rates, we differentiate both sides of the equation above.

\[ \frac{d}{dt}c^2 = \frac{d}{dt} \left( 256 + b^2 + \frac{32}{\sqrt{2}}b \right) \]
\[ 2c \frac{dc}{dt} = 2b \frac{db}{dt} + \frac{32}{\sqrt{2}} \frac{db}{dt} \]
\[ \frac{dc}{dt} = \frac{1}{c} \left( \frac{db}{dt} + \frac{16}{\sqrt{2}} \frac{db}{dt} \right) \]

We are asked for \( \frac{dc}{dt} \) after 1 minute, or when \( b = 4 \text{ km/min} \times 1 \text{ min} = 4 \text{ km} \). We need to solve for \( c \) as well, and we can do that using the cosine law formula above,

\[ c^2 = 256 + (4)^2 + \frac{32}{\sqrt{2}}(4) \]
\[ c = 19.0397 \]

Subbing all the known values into the related rates equation,

\[ \frac{dc}{dt} = \frac{1}{19.0397} \left( (4)(4) + \frac{16}{\sqrt{2}}(4) \right) \]
\[ = 3.2172 \text{ km/min} \]

The plane is moving away from the radar station at a rate of 3.2172 km/min.

21. Water is leaking out of an inverted conical tank at a rate of 12000.0 cubic centimeters per min at the same time that water is being pumped into the tank at a constant rate.

- The tank has height 8.0 meters and the diameter at the top is 5.0 meters.
- The depth of the water is increasing at 28.0 centimeters per minute when the height of the water is 4.0 meters.

Find the rate at which water is being pumped into the tank, in cubic centimeters per minute.
The first challenge in this problem is not to get distracted by the given outflow rate. If you imagine water flowing both into and out of the tank, you will get a net rate of change of the volume, defined by \( \frac{dV}{dt} = \) rate water in \(-\) rate water out, where \( V = \) actual volume of water in the tank. The information about the water level rising is directly tied to the volume, so we focus on that first.

The second point is that we should use consistent units throughout. That will be easier if we standardize on centimeters. Also note that the diameter at the top is 5 m (500 cm), so the radius will be 250 cm.

We are given \( \frac{dh}{dt} \), and want to know \( \frac{dV}{dt} \), so we need a relationship between \( h \) and \( V \) that will always be true. With start with the formula for the volume of a cone:

\[
\text{Volume of water in the tank} = \frac{1}{3} \pi r^2 h
\]

There’s an extra \( r \) in there though, which we need to put in terms of \( h \) or \( V \). The \( h/r \) and 800/250 triangles are similar to each other, so

\[
\frac{r}{h} = \frac{250}{800}
\]

so \( r = \frac{5}{16}h \)

Subbing this into the volume formula gives

\[
V = \frac{1}{3} \pi \left( \frac{5}{16}h \right)^2 h
\]

\[
V = \frac{25}{768} \pi h^3
\]

We can now differentiate both sides with respect to \( t \):

\[
\frac{d}{dt} (V) = \frac{d}{dt} \left( \frac{25}{768} \pi h^3 \right)
\]

\[
\frac{dV}{dt} = \frac{25}{768} \pi \left( 3h^2 \frac{dh}{dt} \right)
\]

Subbing in the known values, \( h = 400 \) cm and \( \frac{dh}{dt} = 28 \) cm/min,

\[
\frac{dV}{dt} = \frac{25}{768} \pi \cdot 3 \cdot (400)^2 \cdot 28
\]

\[
\approx 1,374,000 \text{ cm}^3/\text{min}
\]

The represents the net rate of change of water, taking into account both the inflow and the outflow. Since we are told that the outflow is 12,000 cm³/min,

\[
\frac{1374000 = \text{rate water in} - \frac{12000}{\text{net flow}} \text{rate water out}}{\text{rate in} \approx 1,374,000 + 12,000 = 1,386,000 \text{ cm}^3/\text{min}}
\]

So the rate at which water is being poured in is

rate in \( \approx 1,374,000 + 12,000 = 1,386,000 \text{ cm}^3/\text{min} \)

22. A spherical snowball is melting in such a way that its diameter is decreasing at rate of 0.4 cm/min. At what rate is the volume of the snowball decreasing when the diameter is 17 cm? (Note the answer is a positive number).

The volume formula for spheres relates the two quantities, volume and radius/diameter. Let \( r \) be the radius, \( D \) be the diameter, and \( V \) be the volume.

\[
V = \frac{4}{3} \pi r^3
\]

but \( D = 2r \) or \( r = D/2 \),

so \( V = \frac{4}{3} \pi \left( \frac{D}{2} \right)^3 \)

\[
V = \frac{\pi}{6} D^3
\]

Taking the time derivative of both sides,

\[
\frac{d}{dt} (V) = \frac{\pi}{6} \left( 3D^2 \frac{dD}{dt} \right)
\]

Subbing in the known values, \( \frac{dD}{dt} = -0.4 \) cm/min, and \( D = 17 \) cm,

\[
\frac{dV}{dt} = \frac{\pi}{6} \left( 3(17)^2 \left( -0.4 \right) \right)
\]

\[
\approx 182 \text{ cm}^3/\text{min}
\]
23. The gas law for an ideal gas at absolute temperature \( T \) (in kelvins), pressure \( P \) (in atmospheres), and volume \( V \) is

\[ PV = nRT \]

We note that \( n \) and \( R \) are constants, while \( P, V \) and \( T \) are changing with time. We want to find \( \frac{dT}{dt} \).

Taking the derivative of both sides with respect to time,

\[ \frac{d}{dt}(PV) = \frac{d}{dt}(nRT) \]

\[ \frac{dP}{dt}V + P\frac{dV}{dt} = nR\frac{dT}{dt} \]

\[ \frac{dT}{dt} = \frac{1}{nR} \left( \frac{dP}{dt}V + P\frac{dV}{dt} \right) \]

Filling in the known values from the question,

\[ \frac{dT}{dt} = \frac{1}{(10)(0.0821)} \left((+0.1)(10) + (8)(-0.15)\right) \]

\[ \frac{dT}{dt} \approx -0.2436 \text{ degrees/min} \]

24. The frequency of vibrations of a vibrating violin string is given by \( f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} \), where \( L \) is the length of the string, \( T \) is the tension, and \( \rho \) is its linear density.

Find the rate of change of the frequency with respect to:

(a) the length (when \( T \) and \( \rho \) are constant)
(b) the tension (when \( L \) and \( \rho \) are constant)
(c) the linear density (when \( L \) and \( T \) are constant)

The pitch of a note is determined by the frequency \( f \). (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in (a) through (c) to determine what happens to the pitch of a note:

(d) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates.

(e) when the tension is increased by turning a tuning peg.

(f) when the linear density is increased by switching to another string.

We start with the equality between the original quantities,

\[ PV = nRT \]

We note that \( n \) and \( R \) are constants, while \( P, V \) and \( T \) are changing with time. We want to find \( \frac{dT}{dt} \).

Filling in the known values from the question,

\[ \frac{dT}{dt} = \frac{1}{2L^2} \sqrt{\frac{T}{\rho}} \]

\[ \frac{df}{dL} = \frac{-1}{2L^2} \sqrt{\frac{T}{\rho}} \]

\[ \frac{df}{dT} = \frac{1}{4L\sqrt{\frac{T}{\rho}}} \]

\[ \frac{df}{d\rho} = \frac{-\sqrt{T}}{4L\rho^{3/2}} \]

(d) If we shorten the string length \( L \), we are making \( \Delta L \) negative. Also the derivative \( \frac{df}{dL} \) is negative. Using our linearity relationship, we can estimate \( \Delta f \approx \frac{df}{dL}(\Delta L) \).

Looking at the signs of each element, sign \( \Delta f = (-)(-)(+) \).

Since \( \Delta f \) will be positive, the frequency will increase and the pitch will be higher.

(e) If we increase the string tension \( T \), and the derivative \( \frac{df}{dT} \) is positive, the frequency will be higher, so the pitch will be higher.

(f) If we increase the string density \( \rho \), and the derivative \( \frac{df}{d\rho} \) is negative, the frequency will be lower, so the pitch will be lower.

25. A potter forms a piece of clay into a right circular cylinder. As she rolls it, the height \( h \) of the cylinder increases and the radius \( r \) decreases. Assume that no clay is lost in the process. Suppose the height of the cylinder is increasing by 0.4 centimeters per second. What is the rate at which the radius is changing when the radius is 3 centimeters and the height is 12 centimeters?

The cylinder of clay has a volume of \( V = \pi r^2 h \). We are given \( \frac{dh}{dt} = 0.4 \text{ cm/s} \), and are asked to find \( \frac{dr}{dt} \). We are also told that no clay is lost, so we can also state that the volume is unchanging, or \( \frac{dV}{dt} = 0 \).

Starting with the known relationship \( V = \pi r^2 h \), we differentiate both sides with respect to \( t \) to get the related rates.

\[ \frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h) \]

\[ \frac{dV}{dt} = \pi 2r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt} \]

Subbing \( \frac{dV}{dt} = 0 \), and solving for the rate we want,
\[
\frac{dr}{dt},
0 = \pi 2r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}
\]
\[
2r \frac{dr}{dt} h = -\pi r^2 \frac{dh}{dt}
\]
\[
\frac{dr}{dt} = -\frac{\pi r^2}{2r h}
\]
\[
\frac{dr}{dt} = -\frac{r^2}{2h}
\]

Subbing in the values we know, \( r = 3, h = 12 \) and \( \frac{dh}{dt} = 0.4 \).
\[
\frac{dr}{dt} = -\frac{(3)(0.4)}{2(12)} = -0.05 \text{ cm/s}
\]
The radius of the cylinder of clay is decreasing at 0.05 cm/s at that moment.

26. The volume \( V \) of a right circular cylinder of radius \( r \) and height \( h \) is \( V = \pi r^2 h \).

(a) How is \( \frac{dV}{dt} \) related to \( \frac{dr}{dt} \) if \( h \) is constant and \( r \) varies with time?

(b) How is \( \frac{dV}{dt} \) related to \( \frac{dh}{dt} \) if \( r \) is constant and \( h \) varies with time?

(c) How is \( \frac{dV}{dt} \) related to \( \frac{dh}{dt} \) and \( \frac{dr}{dt} \) if both \( h \) and \( r \) vary with time?

(a) \( 2\pi rh(dr/dt) \)

(b) \( \pi r^2(dh/dt) \)

(c) \( \pi(2rh(dr/dt) + r^2(dh/dt)) \)

27. A hot air balloon rising vertically is tracked by an observer located 2 miles from the lift-off point. At a certain moment, the angle between the observer’s line-of-sight and the horizontal is \( \frac{\pi}{6} \), and it is changing at a rate of 0.1 rad/min. How fast is the balloon rising at this moment?

Let \( y \) be the height of the balloon (in miles) and \( \theta \) the angle between the line-of-sight and the horizontal.

Drawing a sketch with the ground and the balloon above the ground making a right angle, we have \( \tan \theta = \frac{y}{2} \).

Taking the time derivative of both sides,
\[
\frac{d}{dt} (\tan \theta) = \frac{d}{dt} \left( \frac{y}{2} \right)
\]
\[
\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{2} \frac{dy}{dt}
\]
\[
\frac{dy}{dt} = 2 \frac{d\theta}{dt} \sec^2 \theta.
\]

Using \( \frac{d\theta}{dt} = 0.1 \) and \( \theta = \frac{\pi}{6} \) yields
\[
\frac{dy}{dt} = 2(0.1) \frac{1}{\cos^2 \left( \frac{\pi}{6} \right)} \approx 0.2666 \text{ mi/min}.
\]

28. A road perpendicular to a highway leads to a farmhouse located 5 mile away. An car traveling on the highway passes through this intersection at a speed of 55mph.

How fast is the distance between the car and the farmhouse increasing when the car is 7 miles past the intersection of the highway and the road?

Let \( l \) denote the distance between the car and the farmhouse and let \( s \) denote the distance past the intersection of the highway and the road. As seen in the diagram, there is a right-angled triangle made, satisfying \( l^2 = 5^2 + s^2 \).

Taking the derivative of both sides of this equation with respect to \( t \) yields
\[
\frac{d}{dt} (l^2) = \frac{d}{dt} (5^2 + s^2)
\]
\[
2l \frac{dl}{dt} = 2s \frac{ds}{dt}
\]
and so
\[
\frac{dl}{dt} = \frac{s}{l} \frac{ds}{dt}.
\]

We were told in the question that the car is moving at \( \frac{ds}{dt} = 55 \text{ miles/hr} \). When the car is \( s = 7 \) miles past the intersection, we can solve for \( l \) using the Pythagoras relationship we started with, \( l = \sqrt{5^2 + 7^2} = \sqrt{74} \).

This gives \( \frac{dl}{dt} = \frac{7}{\sqrt{74}} \approx 44.7553 \text{ mph} \)

29. Assume that the radius \( r \) of a sphere is expanding at a rate of 6in./min. The volume of a sphere is \( V = \frac{4}{3}\pi r^3 \).

Determine the rate at which the volume is changing with respect to time when \( r = 11 \text{ in} \).
As the radius is expanding at 6 inches per minute, we know that \( \frac{dr}{dt} = 6 \text{ in./min.} \) Taking the derivative with respect to \( t \) of the equation \( V = \frac{4}{3}\pi r^3 \) yields
\[
\frac{dV}{dt} = \frac{4}{3}\pi \left(3r^2 \frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt}
\]
Substituting \( r = 11 \) and \( \frac{dr}{dt} = 6 \) yields \( \frac{dV}{dt} = 4\pi 11^2(6) \approx 9123.19 \text{ in./min.} \).

30. The radius of a circular oil slick expands at a rate of 7 m/min.

(a) How fast is the area of the oil slick increasing when the radius is 26 m?
(b) If the radius is 0 at time \( t = 0 \), how fast is the area increasing after 2 mins?

Let \( r \) be the radius of the oil slick and \( A \) its area.
Then \( A = \pi r^2 \) and \( \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \).
Substituting \( r = 26 \) and \( \frac{dr}{dt} = 7 \), we find \( \frac{dA}{dt} = 2\pi (26)(7) \approx 1143.54 \text{ m}^2/\text{min.} \).

(b) Our formula for the rate of change of area requires us to know \( r \), but here we only know the time \( t \). Our job then is to find out the radius of the oil slick after 2 minutes.

Since the slick’s radius is growing at 7 m/min, and started at \( r = 0 \) at 0 minutes, we can just multiply to find that after 2 minutes, \( r = (7 \text{ m/min}) \times (2 \text{ min}) = 14 \text{ m.} \)

We can use that \( r = 14 \) to compute the rate of area change:
\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi (14)(7) \approx 615.752 \text{m}^2/\text{min.}
\]

31. A searchlight rotates at a rate of 2 revolutions per minute. The beam hits a wall located 10 miles away and produces a dot of light that moves horizontally along the wall. How fast (in miles per hour) is this dot moving when the angle \( \theta \) between the beam and the line through the searchlight perpendicular to the wall is \( \frac{\pi}{6} \)?

Note that \( \frac{d\theta}{dt} = 2(2\pi) = 4\pi \text{ rad/minute.} \)

Here is a diagram showing the arrangement of the elements in the problem. It is a view on the scene from above. As the lighthouse rotates its beam, the angle \( \theta \) will change.

Let \( y \) be the distance between the dot of light and the point of intersection of the wall and the line through the searchlight perpendicular to the wall. Let \( \theta \) be the current angle between the beam of light and the line.

Using trigonometry, we have \( \tan \theta = \frac{y}{10} \).

Differentiating both side with respect to \( t \),
\[
\frac{d}{dt} \tan(\theta) = \frac{d}{dt} \left( \frac{1}{10}y \right)
\]
\[
\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dy}{dt}
\]

We want to find the speed of the light spot along the wall, which is defined by \( \frac{dy}{dt} \), so we solve for that:
\[
\frac{dy}{dt} = 10 \frac{d\theta}{dt} \sec^2 \theta
\]

We can now focus in on the particular scenario in the question:

- \( \theta = \frac{\pi}{6} \) and
- \( \frac{d\theta}{dt} = 4\pi \). From the question, 2 revolutions per minute = 4\pi radians per minute.

Evaluating for \( \frac{dy}{dt} \) now gives:
\[
\frac{dy}{dt} = 10 (4\pi) \frac{1}{\cos^2 (\pi/6)}
\]
\[
= 53.33\pi \approx 167.55 \text{ mi/min}
\]

Converting to miles per hour gives \( \frac{dy}{dt} \approx 10053.09 \text{ mph.} \)

(Yes, this looks crazy fast, but if you imagine yourself watching as the light house beam rotates, it actually can approach an infinite speed as the angle gets close to 90 degrees from the wall.)

32. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?

We note that the kite motion of 8 ft/s represents \( \frac{dx}{dt} \) in this diagram. We are searching for \( \frac{d\theta}{dt} \), so we need
a relationship between $x$ and $\theta$. The simplest is

$$\tan \theta = \frac{100}{x}$$

Taking $\frac{d}{dt}$ of both sides,

$$\frac{d}{dt} (\tan \theta) = \frac{d}{dt} \left( \frac{100}{x} \right)$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{-100}{x^2} \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-100}{x^2 \sec^2 \theta} \frac{dx}{dt}$$

We can use the $L$ value of 200 ft (200 ft of string) with Pythagoras to solve for $x = 173.21$ ft, and $\theta \approx 0.5236$ rad. Subbing those and $\frac{dx}{dt} = 8$ into the related rates equation gives

$$\frac{d\theta}{dt} = \frac{-100 \cos^2 (0.5236)}{173.21^2} \cdot 8$$

$$\frac{d\theta}{dt} = -0.02 \text{ radians/s}$$

33. If two resistors with resistances $R_1$ and $R_2$ are connected in parallel, as in the figure, then the total resistance $R$, measured in Ohms (Ω), is given by:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If $R_1$ and $R_2$ are increasing at rates of $0.3 \text{Ω/s}$ and $0.2 \text{Ω/s}$, respectively, how fast is $R$ increasing when $R_1 = 80 \text{Ω}$ and $R_2 = 100 \text{Ω}$?

![Diagrams](image)

There are several ways to solve this problem. The most straightforward is to take the relationship that is given, and differentiate with respect to $t$.

$$\frac{d}{dt} \left( \frac{1}{R} \right) = \frac{d}{dt} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$-1 \frac{dR}{R^2} = -1 \frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2}$$

Solving for $\frac{dR}{dt}$ gives:

$$\frac{dR}{dt} = R^2 \left( \frac{1}{R_1^2} \frac{dR_1}{dt} + \frac{1}{R_2^2} \frac{dR_2}{dt} \right)$$

In the problem, we are given the following values:

- $R_1 = 80$ and $R_2 = 100$, and
- $\frac{dR_1}{dt} = +0.3$ and $\frac{dR_2}{dt} = 0.2$.

This covers all the values in our expression for $\frac{dR}{dt}$, except for $R$ itself. To find that, we can use the original equation to solve for overall resistance:

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{100}$$

$$\frac{1}{R} = \frac{9}{400}$$

so $R = \frac{400}{9} \approx 44.444 \text{ ohms}$

Substituting in all these values into our $\frac{dR}{dt}$ expression gives

$$\frac{dR}{dt} = R^2 \left( \frac{1}{80^2} (0.3) + \frac{1}{100^2} (0.2) \right)$$

$$\frac{dR}{dt} = 0.132 \text{ ohms/s}$$

34. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at a rate of 0.2 m$^3$/min, how fast is the water level rising when the water is 30 cm deep?

![Diagrams](image)

Diagrams are essential for this problem!
When water is filling up the trough, we have a connection between the filled depth \( h \), and the volume \( V \).

We are given that \( \frac{dV}{dt} = 0.2 \text{ m}^3/\text{min} \), and we want \( \frac{dh}{dt} \) when \( h = 30 \text{ cm} \).

We need an equation that relates \( h \) and \( V \) at any time during the filling process.

The depth of the trough is 10 m (not shown in the diagrams), so

\[
V = 10 \times (\text{cross-section area up to height } h)
\]

The cross-section area in the diagram is made up of three elements: \( T_1 \), \( R \) and \( T_2 \). If we express everything in terms of meters (so 30 cm = 0.3 m),

\[
R = 0.3h
\]

The areas of \( T_1 \) and \( T_2 \) can be determined using similar triangles, as their height/widths are in a 50 cm/25 cm ratio: \( h/\text{width} = 0.50/0.25 \), so width = 0.5h.

This means that (area of \( T_1 \) = (area of \( T_2 \) = \( \frac{1}{2}hw = \frac{1}{2}h(0.5h) = \frac{1}{4}h^2 \).

Bringing it all back into the volume expression,

\[
V = 10 \times (\text{cross-section area up to height } h) \\
= 10(T_1 + R + T_2) \\
= 10 \left( \frac{1}{4}h^2 + 0.3h + \frac{1}{4}h^2 \right) \\
= 10 \left( \frac{1}{2}h^2 + 0.3h \right) \\
= 5h^2 + 3h
\]

We can now differentiate both sides with respect to time to find the relationship between \( \frac{dV}{dt} \) and \( \frac{dh}{dt} \):

\[
\frac{dV}{dt} = 10h \frac{dh}{dt} + 3 \frac{dh}{dt}
\]

Subbing in \( \frac{dV}{dt} = 0.2 \text{ m}^3/\text{min} \), and \( h = 0.3 \text{ m} \),

\[
0.2 = 10(0.3) \frac{dh}{dt} + 3 \frac{dh}{dt} \\
0.2 = 6 \frac{dh}{dt} \\
\frac{dh}{dt} = 0.033 \text{ m/min}, \text{ or } \\
\frac{dh}{dt} = 100 \times \frac{0.033}{0.3} = 10 \approx 3.33 \text{ cm/min}.
\]

35. A voltage \( V \) across a resistance \( R \) generates a current \( I = V/R \). If a constant voltage of 4 volts is put across a resistance that is increasing at a rate of 0.7 ohms per second when the resistance is 7 ohms, at what rate is the current changing?

We know \( dR/dt = 0.7 \) when \( R = 5 \) and \( V = 9 \) and we want to know \( dI/dt \). Differentiating \( I = V/R \) with \( V \) constant gives

\[
\frac{dI}{dt} = V \left( -\frac{1}{R^2} \frac{dR}{dt} \right),
\]

so substituting gives

\[
\frac{dI}{dt} = 4 \left( -\frac{1}{7^2} \cdot 0.7 \right) \approx -0.0571 \text{ amp/s}.
\]

(Note that we know the units are amp/s from the Leibnitz form of the derivative, \( \frac{dI}{dt} \). It is, of course, equally correct to use the expression on the right-hand side to determine the units, \( \frac{V}{\text{ohm}} \left( \frac{\text{ohm}}{\text{s}} \right) \), or, \( V/\text{ohm} \text{*s} \).)