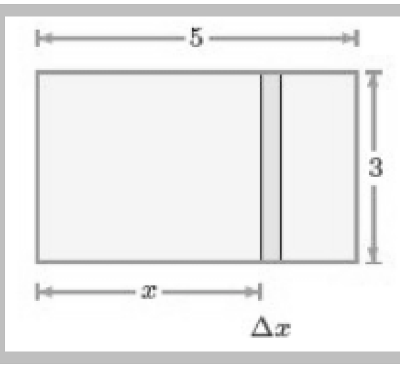
Unit #13 - Integration to Find Areas and Volumes, Volumes of Revolution

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Areas

In Questions #1-8, find the area of one strip or slice, then use that to build a definite integral representing the total area of the region. Where possible with the techniques from the class, evaluate the integral.

1.



Each slice has area 3 Δx, so the total area, using n intervals, is given by

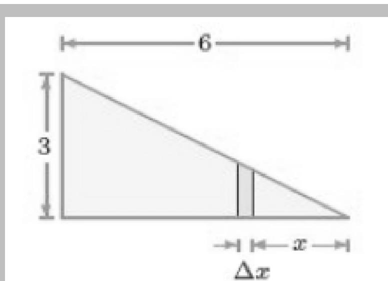
$$\text{Total area} \approx \sum_{i=1}^{n} 3\ \Delta x_i$$

Taking the limit as $n \to \infty$ or $\Delta x \to 0$, and noting that the first slice is at $x = 0$ and the last slice is at $x = 5$,

$$\begin{aligned}\text{Total area} &= \int_{x=0}^{x=5} 3\ dx \\ &= 3x\Big|_0^5 = 3(5-0) = 15 \text{ sq. units}\end{aligned}$$

Check: this matches the area of a 3×5 rectangle computed using (width)×(height).

2.



Each slice will be roughly rectangular in shape, with height y and width Δx. From similar triangles, the height y for a slice a position x will satisfy

$$\begin{aligned}\frac{y}{3} &= \frac{x}{6} \\ \text{or } y &= \frac{x}{2}\end{aligned}$$

Thus the area of one rectangular slice will be

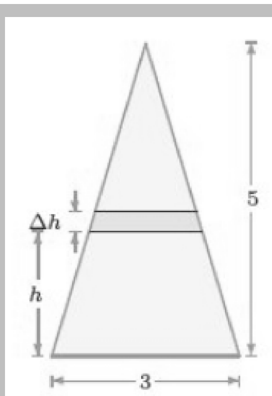
$$\begin{aligned} \text{width} \times \text{height} &= (\Delta x)\frac{x}{2} \\ &= \frac{x}{2}(\Delta x) \end{aligned}$$

Since the slices begin at $x = 0$ and end at $x = 6$,

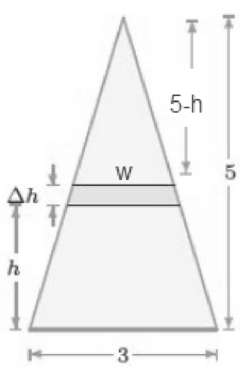
$$\begin{aligned} \text{Total area} &= \int_0^6 \frac{x}{2}\, dx \\ &= \frac{x^2}{4}\bigg|_0^6 = \frac{36}{4} - 0 = 9 \text{ sq. units} \end{aligned}$$

Check: Area of a triangle $= \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}3 \cdot 6 = 9$ sq. units.

3.



Each slice will be roughly rectangular in shape, with width w and height Δh.
There is a similar triangle with height $5 - h$ which is similar to the given triangle.



Based on this diagram, the width w for a slice a position h, given h as shown, will satisfy

$$\begin{aligned} \frac{w}{5-h} &= \frac{3}{5} \\ \text{or } w &= \frac{15 - 3h}{5} \end{aligned}$$

Thus the area of one rectangular slice will be

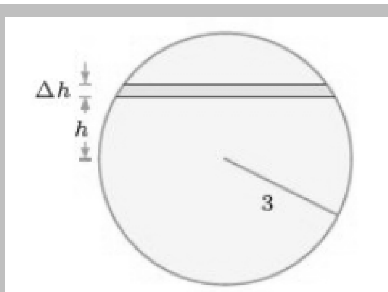
$$\text{width} \times \text{height} = \frac{15 - 3h}{5}(\Delta h)$$

Since the slices begin at $h = 0$ (bottom) and end at $h = 5$ (top),

$$\begin{aligned}
\text{Total area} &= \int_0^5 \frac{15 - 3h}{5}\, dh \\
&= \frac{1}{5}\left(15h - \frac{3h^2}{2}\right)\Big|_0^5 \\
&= \frac{1}{5}\left(15(5) - \frac{3(5^2)}{2}\right) = 7.5 \text{ sq. units}
\end{aligned}$$

Check: Area of a triangle $= \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2} 3 \cdot 5 = 7.5$ sq. units.

4.



Note: your final integral will **not** be of a type you can evaluate using substitution or by parts. Stop once you have created the integral.

Suppose the length of the strip shown is w. Then the Pythagorean theorem gives

$$h^2 + \left(\frac{w}{2}\right)^2 = 3^2 \quad \text{so} \quad w = 2\sqrt{3^2 - h^2}.$$

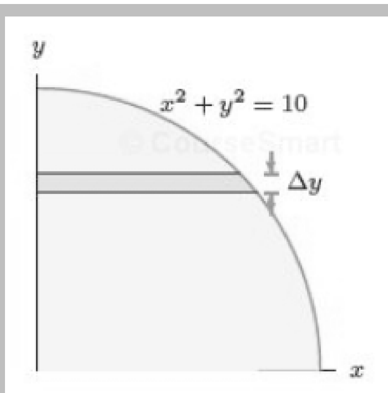
Thus

$$\begin{aligned}
\text{Area of strip } &\approx w\Delta h = 2\sqrt{3^2 - h^2}\,\Delta h, \\
\text{Area of region } &= \int_{-3}^{3} 2\sqrt{3^2 - h^2}\, dh.
\end{aligned}$$

Note that this integral is not one we've seen before (squared h^2 term inside the square root). This would not naturally be a substitution integral (no outside h to cancel), and through some trial and error we find that it is not simplified by using integration by parts either. This is an integral we cannot actually evaluate this integral with the integral techniques covered so far.

On a test or exam, we would make it clear that you could stop after writing the integral.

5.



Note: your final integral will **not** be of a type you can evaluate using substitution or by parts. Stop once you have created the integral.

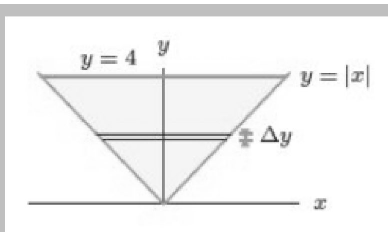
The strip has width Δy, so the variable of integration is y. The length of the strip is x. Since $x^2 + y^2 = 10$ and the region is in the first quadrant, solving for x gives $x = \sqrt{10 - y^2}$. Thus

$$\text{Area of strip } \approx x\Delta y = \sqrt{10 - y^2}\, dy.$$

The region stretches from $y = 0$ to $y = \sqrt{10}$, so

$$\text{Area of region } = \int_0^{\sqrt{10}} \sqrt{10 - y^2}\, dy.$$

6.



The strip has width Δy, so the variable of integration is y. The length of the strip is $2x$ for $x \geq 0$. For positive x, we have $x = y$. Thus,

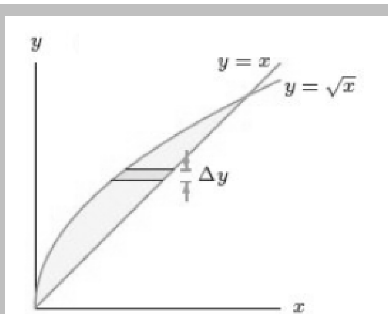
$$\text{Area of strip } \approx 2x\Delta y = 2y\Delta y.$$

Since the region extends from $y = 0$ to $y = 4$,

$$\text{Area of region } = \int_0^4 2y\, dy = y^2\Big|_0^4 = 16.$$

Check: The area of the region can be computed by $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2} \cdot 8 \cdot 4 = 16$.

7.



The width of the strip is Δy, so the variable of integration is y. Since the graphs are $x = y$ and $x = y^2$, the length of the strip is $y - y^2$, and

$$\text{Area of strip } \approx (y - y^2)\Delta y.$$

The curves cross at the points $(0, 0)$ and $(1, 1)$, so

$$\text{Area of region } = \int_0^1 (y - y^2)\, dy = \frac{y^2}{2} - \frac{y^3}{3}\bigg|_0^1 = \frac{1}{6}.$$

8.

The width of the strip is Δx, so the variable of integration is x. The line has equation $y = 6 - 3x$. The length of the strip is $6 - 3x - (x^2 - 4) = 10 - 3x - x^2$. (Since $x^2 - 4$ is negative where the graph is below the x-axis, subtracting $x^2 - 4$ there adds the length below the x-axis.) Thus

$$\text{Area of strip } \approx (10 - 3x - x^2)\Delta x.$$

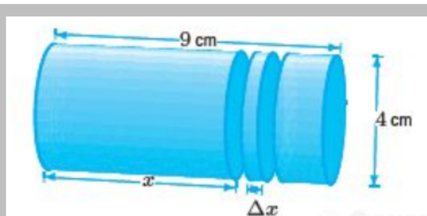
Both graphs cross the x-axis where $x = 2$, so

$$\text{Area of region } = \int_0^2 (10 - 3x - x^2)\, dx = 10x - \frac{3}{2}x^2 - \frac{x^3}{3}\bigg|_0^2 = \frac{34}{3}.$$

Volumes of Geometric Shapes

In Questions #9-14, write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown. Evaluate the integral exactly. (Regions are parts of cones, cylinders, spheres, and pyramids.)

9.



9. Each slice is a circular disk with radius $r = 2$ cm.

$$\text{Volume of disk } = \pi r^2 \Delta x = 4\pi \Delta x \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum 4\pi \Delta x \text{ cm}^3.$$

Taking a limit as $\Delta x \to 0$, we get

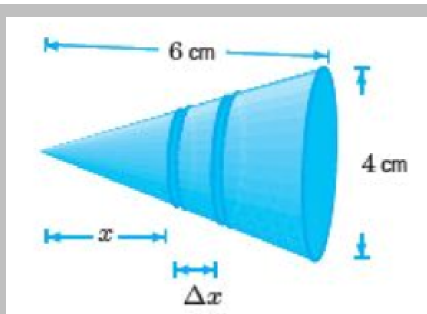
$$\text{Total volume } = \lim_{\Delta x \to 0} \sum 4\pi \Delta x = \int_0^9 4\pi \, dx \text{ cm}^3.$$

Evaluating gives

$$\text{Total volume } = 4\pi x \Big|_0^9 = 36\pi \text{ cm}^3.$$

Check: The volume of the cylinder can also be calculated using the formula $V = \pi r^2 h = \pi 2^2 \cdot 9 = 36\pi$ cm^3.

10.



Each slice is a circular disk. Since the radius of the cone is 2 cm and the length is 6 cm, the radius is one-third of the distance from the vertex. Thus, the radius at x is $r = x/3$ cm. See Figure 8.1.

$$\text{Volume of slice } \approx \pi r^2 \Delta x = \frac{\pi x^2}{9} \Delta x \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum \pi \frac{x^2}{9} \Delta x \text{ cm}^3.$$

Taking a limit as $\Delta x \to 0$, we get

$$\text{Total volume } = \lim_{\Delta x \to 0} \sum \pi \frac{x^2}{9} \Delta x = \int_0^6 \pi \frac{x^2}{9} \, dx \text{ cm}^3.$$

Evaluating, we get

$$\text{Total volume } = \frac{\pi}{9} \frac{x^3}{3} \Big|_0^6 = \frac{\pi}{9} \cdot \frac{6^3}{3} = 8\pi \text{ cm}^3.$$

Check: The volume of the cone can also be calculated using the formula $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 2^2 \cdot 6 = 8\pi$ cm^3.

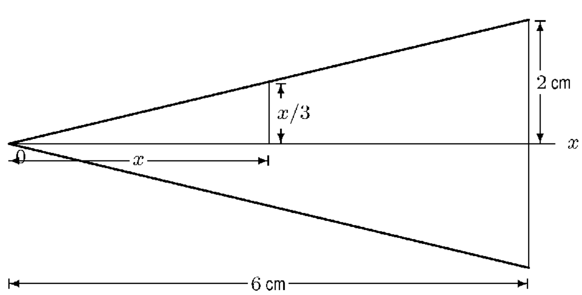
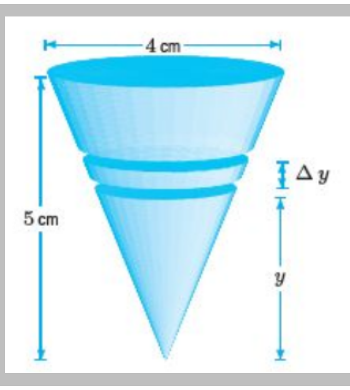


Figure 8.1

11.



Each slice is a circular disk. From Figure 8.2, we see that the radius at height y is $r = \frac{2}{5}y$ cm. Thus

$$\text{Volume of disk} \approx \pi r^2 \Delta y = \pi \left(\frac{2}{5}y\right)^2 \Delta y = \frac{4}{25}\pi y^2 \Delta y \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume} \approx \sum \frac{4\pi}{25} y^2 \Delta y \text{ cm}^3.$$

Taking the limit as $\Delta y \to 0$, we get

$$\text{Total volume} = \lim_{\Delta y \to 0} \sum \frac{4\pi}{25} y^2 \Delta y = \int_0^5 \frac{4\pi}{25} y^2 \, dy \text{ cm}^3.$$

Evaluating gives

$$\text{Total volume} = \frac{4\pi}{25} \frac{y^3}{3}\bigg|_0^5 = \frac{20}{3}\pi \text{ cm}^3.$$

Check: The volume of the cone can also be calculated using the formula $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}2^2 \cdot 5 = \frac{20}{3}\pi$ cm^3.

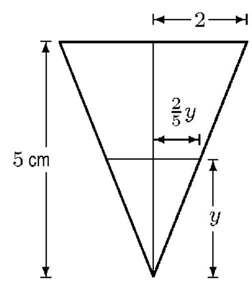


Figure 8.2

12. Note: for this example, the integral you construct cannot be evaluated using just the techniques from class. **However**, if you think of area interpretations of the integral, you can still evaluate the integral.

Each slice is a rectangular slab of length 10 m and width that decreases with height. See Figure 8.3. At height y, the length x is given by the Pythagorean Theorem

$$y^2 + x^2 = 7^2.$$

Solving gives $x = \sqrt{7^2 - y^2}$ m. Thus the width of the slab is $2x = 2\sqrt{7^2 - y^2}$ and

$$\text{Volume of slab} = \text{Length} \cdot \text{Width} \cdot \text{Height} = 10 \cdot 2\sqrt{7^2 - y^2} \cdot \Delta y = 20\sqrt{7^2 - y^2}\Delta y \text{ m}^3.$$

Summing over all slabs, we have

$$\text{Total volume} \approx \sum 20\sqrt{7^2 - y^2}\Delta y \text{ m}^3.$$

Taking a limit as $\Delta y \to 0$, we get

$$\text{Total volume} = \lim_{\Delta y \to 0} \sum 20\sqrt{7^2 - y^2}\Delta y = \int_0^7 20\sqrt{7^2 - y^2}\, dy \text{ m}^3.$$

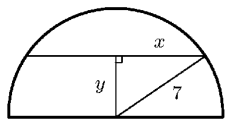


Figure 8.3

13.

Each slice is a circular disk. See Figure 8.4. The radius of the sphere is 5 mm, and the radius r at height y is given by the Pythagorean Theorem

$$y^2 + r^2 = 5^2.$$

Solving gives $r = \sqrt{5^2 - y^2}$ mm. Thus,

$$\text{Volume of disk} \approx \pi r^2 \Delta y = \pi(5^2 - y^2)\Delta y \text{ mm}^3.$$

Summing over all disks, we have

$$\text{Total volume} \approx \sum \pi(5^2 - y^2)\Delta y \text{ mm}^3.$$

Taking the limit as $\Delta y \to 0$, we get

$$\text{Total volume} = \lim_{\Delta y \to 0} \sum \pi(5^2 - y^2)\Delta y = \int_0^5 \pi(5^2 - y^2)\, dy \text{ mm}^3.$$

Evaluating gives

$$\text{Total volume} = \pi \left(25y - \frac{y^3}{3}\right)\Bigg|_0^5 = \frac{250}{3}\pi \text{ mm}^3.$$

Check: The volume of a hemisphere can be calculated using the formula $V = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi 5^3 = \frac{250}{3}\pi$ mm^3.

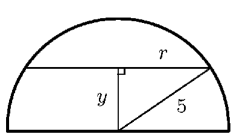
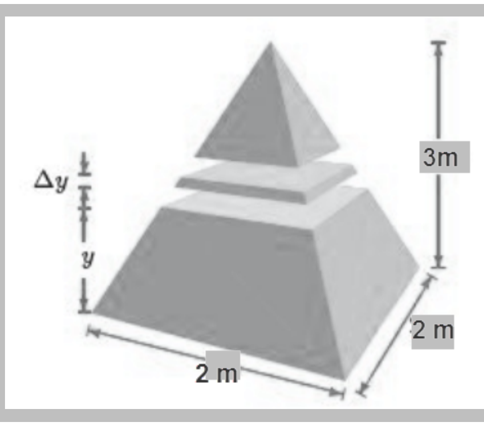


Figure 8.4

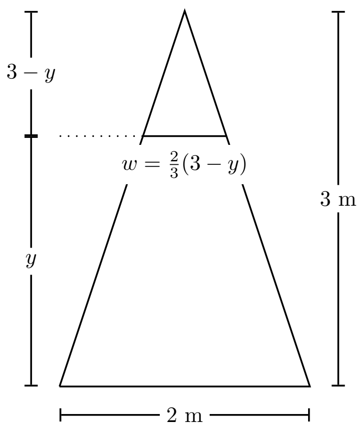
14.



Each horizontal slice is a square; the side length decreases as we go up the pyramid.

By using similar triangles (large 2 wide, 3 high; small w wide, $(3-y)$ high), we can find the width of a slice at height y:

$$\frac{w}{(3-y)} = \frac{2}{3}$$
$$\text{so } w = \frac{2}{3}(3-y)$$



The volume of each slice is a thin square parallelpiped, with volume

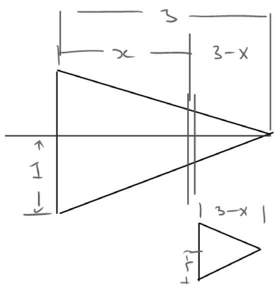
$$\text{slice vol } = (\text{width})^2 \times \ (\text{thickness}) = w^2 \times \Delta y$$

The total volume is then the sum of all the slice volumes, with $\Delta y \to dy$:

$$\begin{aligned}
\text{Total volume } &= \int_{y=0}^{y=3} w^2 \ dy \\
&= \int_{y=0}^{y=3} \left(\frac{2}{3}(3-y)\right)^2 \ dy \\
&= \left(\frac{4}{9}\right) \frac{-(3-y)^3}{3}\bigg|_0^3 \\
&= \left(\frac{4}{9}\right) \frac{1}{3}\left(-0-3^3\right) \\
&= 4 \text{ m}^3
\end{aligned}$$

Check: the volume of any pyramid/cone can be computed using the formula $V = \dfrac{1}{3} A_{\text{base}} h = \dfrac{1}{3} 2^2 \cdot 3 = 4 \text{ m}^3$. This matches our integral calculation.

15. Find, by slicing, the volume of a cone whose height is 3 cm and whose base radius is 1 cm.



$$\frac{r}{3-x} = \frac{1}{3}$$

In the diagram, using the ratios of similar triangles, we see that the radius of a thin slice at the point x from the left side/base of the the cone will satisfy

$$\begin{aligned} \frac{r}{3-x} &= \frac{1}{3} \\ \text{so } r &= \frac{3-x}{3} \\ &= 1 - \frac{x}{3} \end{aligned}$$

The volume of a thin slice Δx thick will then be

$$\text{Slice volume } = \pi r^2 \, \Delta x = \pi \left(1 - \frac{x}{3}\right)^2 \, \Delta x$$

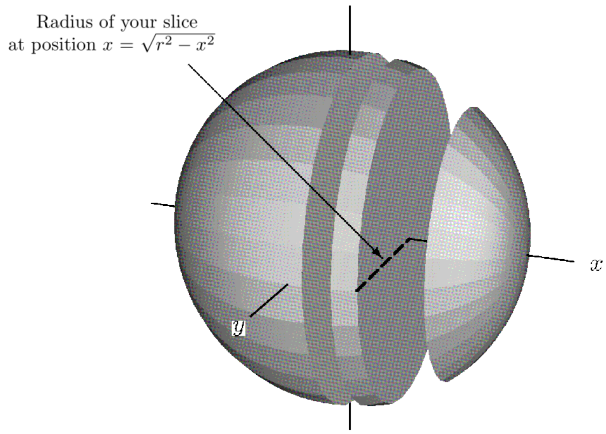
Using an integral to sum the volume in all the slices,

$$\text{Total volume} = \int_{x=0}^{x=3} \pi \left(1 - \frac{x}{3}\right)^2 \, dx$$

Evaluating the integral,

$$\begin{aligned} \text{Total volume} &= \int_0^3 \pi \left(1 - \frac{x}{3}\right)^2 \, dx \\ &= \pi \int_0^3 \left(1 - \frac{2}{3}x + \frac{1}{9}x^2\right) dx \\ &= \pi \left(x - \frac{2}{3}\frac{x^2}{2} + \frac{1}{9}\frac{x^3}{3}\right)\Big|_0^3 \\ &= \pi \left[\left(3 - \frac{1}{3}\,9 + \frac{27}{27}\right) - (0 - 0 + 0)\right] \\ &= \pi\,[1] = \pi \text{ cubic units.} \end{aligned}$$

16. Find the volume of a sphere of radius r by slicing.



We can slice up the sphere several ways; in the diagram we choose to do so by rotating the circle $x^2+y^2 = r^2$ around the x-axis, and then taking vertical/perpendicular slices. Careful with your notation here: r is the radius of the **entire sphere** and **not** the radius of a single slice!
Each slice is a circle, with the **slice** radius defined by the y distance to the circle. With the slice radius $= y = \sqrt{r^2 - x^2}$, we get a sliice **volume** of

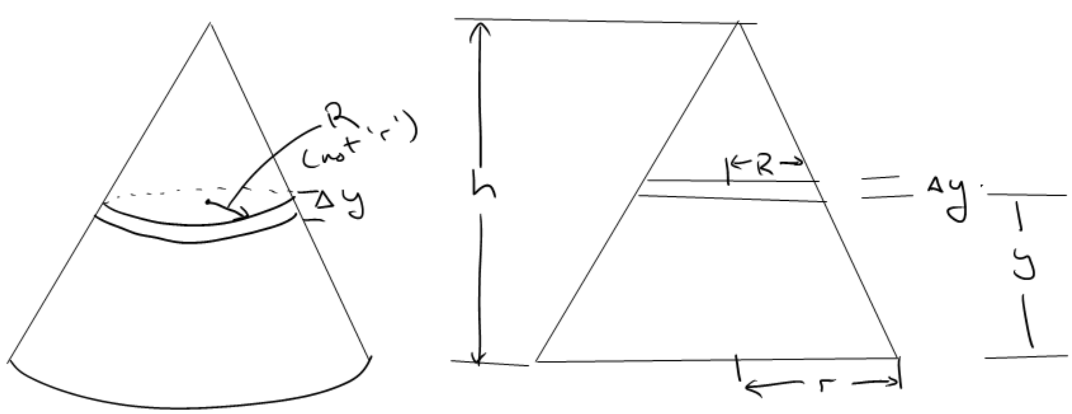
$$\text{slice vol } = \pi(\text{slice radius})^2 = \pi(\sqrt{r^2-x^2})^2 = \pi(r^2 - x^2)$$

This gives the overall volume as

$$\begin{aligned}
V &= \int_{x=-r}^{x=r} \pi(r^2 - x^2)\ dx \\
&= \pi\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r} \\
&= \pi\left(r^2(r) - \frac{r^3}{3}\right) - \pi\left(r^2(-r) - \frac{(-r)^3}{3}\right) \\
&= \pi\left(\frac{2}{3}r^3\right) - \pi\left(\frac{-2}{3}r^3\right) \\
&= \frac{4}{3}\pi r^3, \text{ or the famous formula for the volume of a sphere.}
\end{aligned}$$

17. Find, by slicing, a formula for the volume of a cone of height h and base radius r.

The crucial step in this problem is to define new variables that describe the size of slices of the cone: we can't use r and h because they are already fixed as the height and radius of **entire** cone. In the diagram below, we define the slice location as y vertically and which produces a slice radius of capital R.



We want to find the volume of a generic slice Δy thick, so we need a formula for the slice radius R. To find R, we use similar triangles as in earlier problems, we can find that

- the small triangle at the top (above the slice) has bottom radius R and height $h - y$ (whole height is h, and we've eliminated the bottom y part); and
- the large triangle making the entire cone has bottom radius r and height h.

Using ratios of the sides of these triangles,

$$\frac{R}{r} = \frac{h-y}{h}$$
$$\text{so } R = \frac{r(h-y)}{h}$$

Each slice is a cylinder with radius R and thickness Δy, so

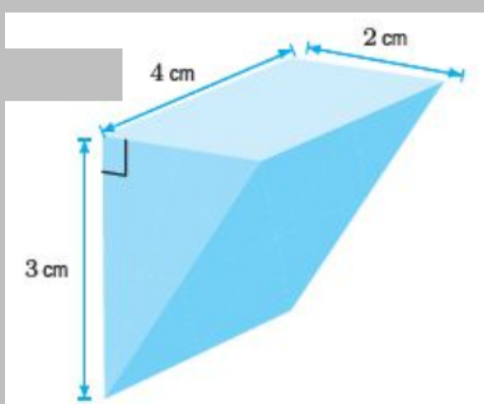
$$\text{slice volume} = \pi R^2 \ \Delta y$$

The **entire** volume can be computed by integrating (adding) the slice volumes up from the bottom ($y = 0$) to the tip of the cone ($y = h$):

$$\begin{aligned}
V &= \int_0^h \pi R^2 \ \Delta y \\
&= \int_0^h \pi \left(\frac{r(h-y)}{h}\right)^2 \ \Delta y \\
\text{factoring out constants:} \quad &= \pi \frac{r^2}{h^2} \int_0^h (h-y)^2 \ \Delta y \\
\text{integrating:} \quad &= \pi \frac{r^2}{h^2} \frac{(h-y)^3}{3}(-1)\Big|_0^h \\
&= \frac{-\pi}{3} \frac{r^2}{h^2} \left[\underbrace{(h-h)^3}_{0} - \underbrace{(h-0)^3}_{h^3}\right] \\
&= +\pi \frac{r^2 h}{3}
\end{aligned}$$

which just happens to perfectly match our known formula for the volume of a cone.

18. The figure below shows a solid with both rectangular and triangular cross sections.



(a) Slice the solid parallel to the triangular faces. Sketch one slice and calculate its volume in terms of x, the distance of the slice from one end. Then write and evaluate an integral giving the volume of the solid.

(b) Repeat part (a) for horizontal slices. Instead of x, use h, the distance of a slice from the top.

(a) A vertical slice has a triangular shape and thickness Δx. See Figure 8.15.

$$\text{Volume of slice} = \text{Area of triangle} \cdot \Delta x = \frac{1}{2} \text{Base} \cdot \text{Height} \cdot \Delta x = \frac{1}{2} \cdot 2 \cdot 3\Delta x = 3\Delta x \text{ cm}^3.$$

Thus,

$$\text{Total volume} = \lim_{\Delta x \to 0} \sum 3\Delta x = \int_0^4 3\,dx = 3x\Big|_0^4 = 12 \text{ cm}^3.$$

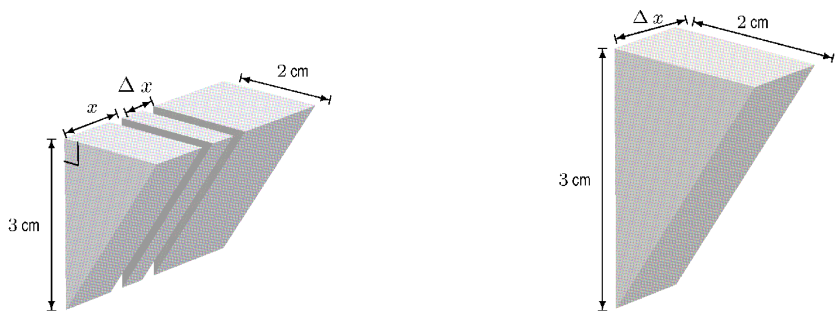


Figure 8.15

(b) A horizontal slice has a rectangular shape and thickness Δh. See Figure 8.16. Using similar triangles, we see that

$$\frac{w}{2} = \frac{3-h}{3},$$

so

$$w = \frac{2}{3}(3-h) = 2 - \frac{2}{3}h.$$

Thus

$$\text{Volume of slice } \approx 4w\Delta h = 4\left(2 - \frac{2}{3}h\right)\Delta h = \left(8 - \frac{8}{3}h\right)\Delta h.$$

So,

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum \left(8 - \frac{8}{3}h\right)\Delta h = \int_0^3 \left(8 - \frac{8}{3}h\right) dh = \left(8h - \frac{4h^2}{3}\right)\Big|_0^3 = 12 \text{ cm}^3.$$

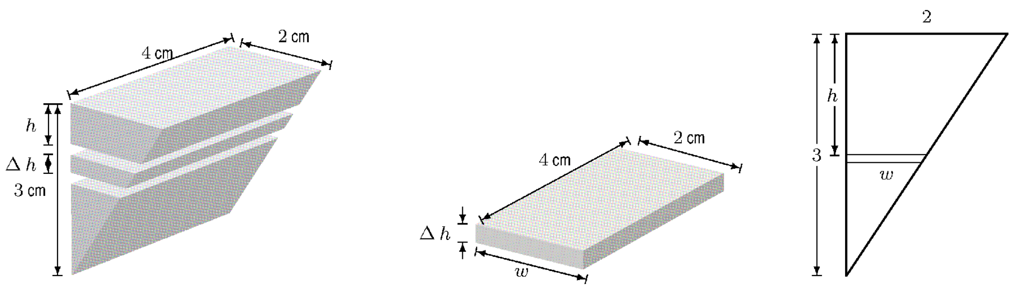
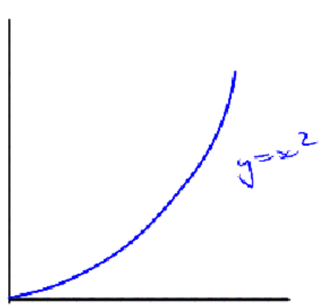


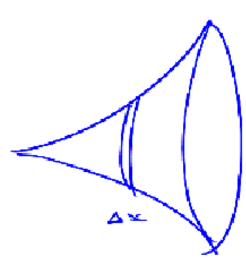
Figure 8.16

Volumes of Revolution

In Questions #19-23, the region is rotated around the x-axis. Find the volume.

19. Bounded by $y = x^2$, $y = 0$, $x = 0$, $x = 1$.





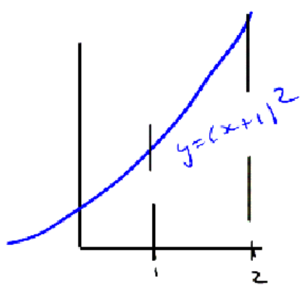
Radii of the slices are given by the y coordinate on the graph of $y = x^2$.

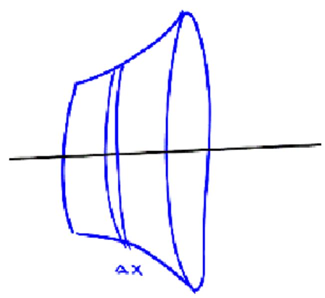
$$\begin{aligned}\text{volume of one slice} &= (\text{circle area})(\text{thickness}) \\ &= (\pi r^2)(\Delta x) \\ &= (\pi (x^2)^2)(\Delta x) \\ \text{Total volume} &= \int_0^1 (\pi (x^2)^2)\ dx\end{aligned}$$

Computing the integral gives

$$\text{Total volume } = \int_0^1 \pi x^4\ dx = \pi \frac{x^5}{5}\Big|_0^1 = \frac{\pi}{5}$$

20. Bounded by $y = (x+1)^2$, $y = 0$, $x = 1$, $x = 2$.





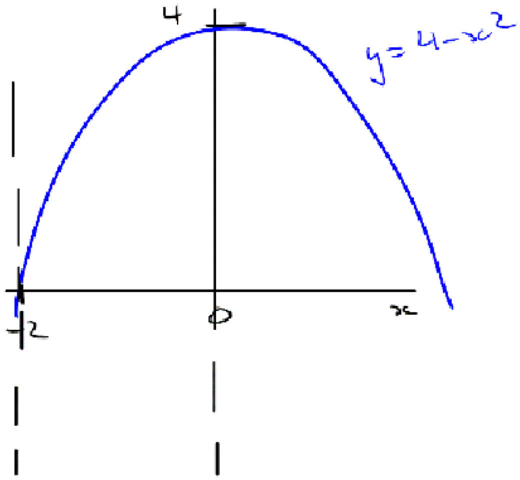
Radii of the slices are given by the y coordinate on the graph of $y = (x+1)^2$.

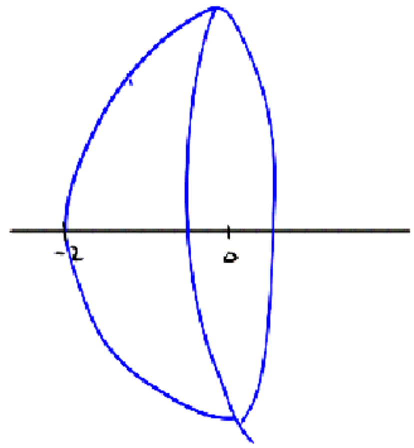
$$\begin{aligned}\text{volume of one slice} &= (\text{circle area})(\text{thickness}) \\ &= (\pi r^2)(\Delta x) \\ &= \pi((x+1)^2)^2(\Delta x) \\ \text{Total volume} &= \int_0^1 \pi((x+1)^2)^2\ dx \\ &= \int_0^1 \pi(x+1)^4\ dx\end{aligned}$$

Computing the integral gives

$$\text{Total volume } = \int_1^2 \pi(x+1)^4\ dx = \pi \frac{(x+1)^5}{5}\Big|_1^2 = \frac{211\pi}{5}$$

21. Bounded by $y = 4 - x^2$, $y = 0$, $x = -2$, $x = 0$.





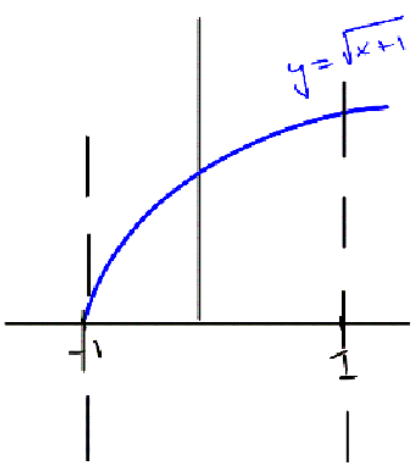
Radii of the slices are given by the y coordinate on the graph of $y = 4 - x^2$.

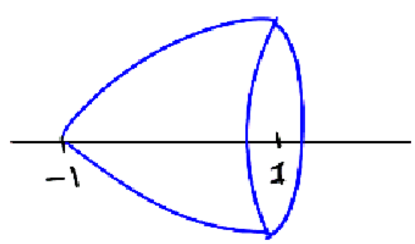
$$\begin{aligned} \text{volume of one slice} &= (\text{circle area})(\text{thickness}) \\ &= (\pi r^2)(\Delta x) \\ &= \pi(4 - x^2)^2(\Delta x) \\ \text{Total volume} &= \int_{-2}^{0} \pi(4 - x^2)^2 \, dx \end{aligned}$$

Computing the integral gives

$$\text{Total volume } = \int_{-2}^{0} \pi \underbrace{(4 - x^2)^2}_{\text{expand}} \, dx = \pi \int (16 - 8x^2 + x^4) \, dx = \pi \left(16x - 8\frac{x^3}{3} + \frac{x^5}{5}\right) \Big|_{-2}^{0} = \frac{256\pi}{15}$$

22. Bounded by $y = \sqrt{x + 1}$, $y = 0$, $x = -1$, $x = 1$.

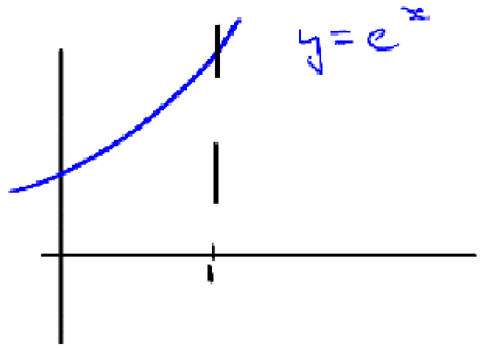


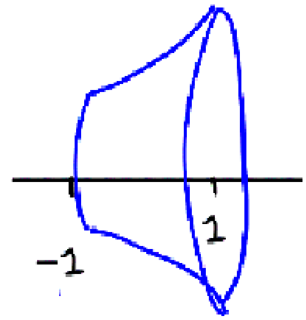


The volume is given by

$$\text{Total volume} = \int_{-1}^{1} \pi(\sqrt{x + 1})^2 \, dx = \pi \int_{-1}^{1} (x + 1) \, dx = \pi \left(\frac{x^2}{2} + x\right) \Big|_{-1}^{1} = 2\pi$$

23. Bounded by $y = e^x$, $y = 0$, $x = -1$, $x = 1$.

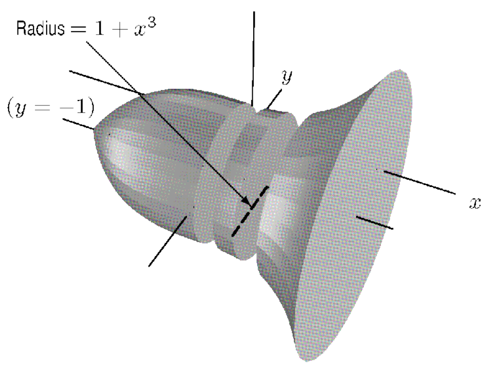




The volume is given by

$$\text{Total volume} = \int_{-1}^{1} \pi (e^x)^2\ dx = \pi \int_{-1}^{1} e^{2x}\ dx = \pi\left(\frac{e^{2x}}{2}\right)\Big|_{-1}^{1} = \frac{\pi}{2}(e^2 - e^{-2})$$

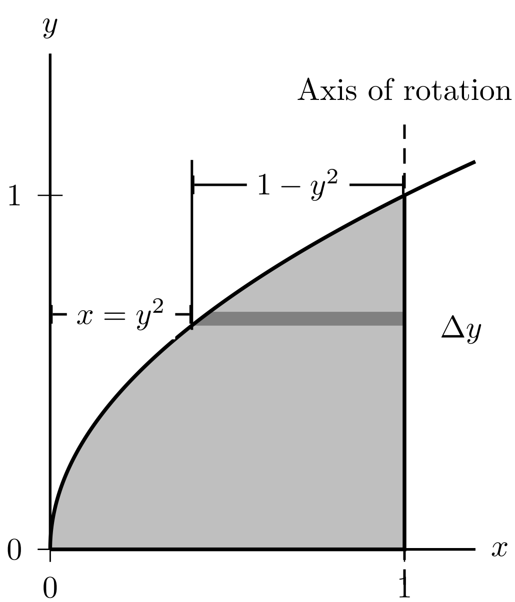
24. Find the volume obtained when the region bounded by $y = x^3$, $x = 1$, $y = -1$ is rotated around the axis $y = -1$.



We slice the region perpendicular to the x–axis. The Riemann sum we get is $\sum \pi(x^3+1)^2 \Delta x$. So the volume V is the integral

$$\begin{aligned} V &= \int_{-1}^{1} \pi (x^3+1)^2\, dx \\ &= \pi \int_{-1}^{1} (x^6 + 2x^3 + 1)\, dx \\ &= \pi\left(\frac{x^7}{7} + \frac{x^4}{2} + x\right)\Big|_{-1}^{1} \\ &= (16/7)\pi \approx 7.18. \end{aligned}$$

25. Find the volume obtained when the region bounded by $y = \sqrt{x}$, $x = 1$, $y = 0$ is rotated around the axis $x = 1$.



Rotating around the $x = 1$ line will require **horizontal** slices, Δy or dy thick.
The shape of each slice will be a disc or thin cylinder, with volume $V = \pi r^2 \Delta y$.
From the diagram, we note that the boundary is defined by $y = \sqrt{x}$, or (re-writen) $x = y^2$. However, because the axis of rotation is **not** a regular axis, we need to be careful about how we use this boundary.

- $x = y^2$ means that the distance from $x = 0$ to the boundary is y^2.
- The distance then between the axis of rotation $x = 1$ and the boundary will $r = 1 - x = 1 - y^2$.

Using this radius, we can build the integral representing the volume of the rotated region.

$$\int_{y=0}^{y=1} \pi(\underbrace{1-y^2}_{r})^2\ dy$$

26. Find the volume obtained when the region bounded by $y = x^2$, $y = 1$, and the y-axis is rotated around the y-axis.

Slice the object into disks horizontally, as in Figure 8.23. A typical disk has thickness Δy and radius $x = \sqrt{y}$. Thus

$$\text{Volume of slice } \approx \pi x^2 \Delta y = \pi y\, \Delta y.$$

$$\text{Volume of solid } = \lim_{\Delta y \to 0} \sum \pi y\, \Delta y = \int_0^1 \pi y\, dy = \pi \frac{y^2}{2}\bigg|_0^1 = \frac{\pi}{2}.$$

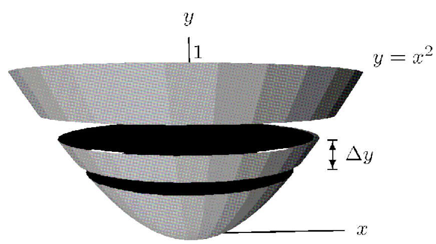


Figure 8.23

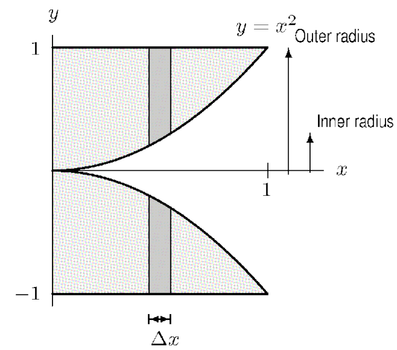


Figure 8.24: Cross-section of solid

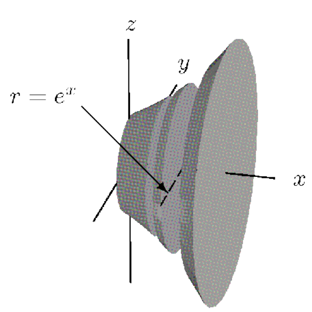
27. Find the volume obtained when the region in the 1st (upper-right) quadrant, bounded by $y = x^2$, $y = 1$, and the y-axis, is rotated around the x-axis.

Slice the object into rings vertically, as is Figure 8.24. A typical ring has thickness Δx and outer radius $y = 1$ and inner radius $y = x^2$.

$$\text{Volume of slice } \approx \pi 1^2 \Delta x - \pi y^2 \Delta x = \pi(1 - x^4)\, \Delta x.$$

$$\text{Volume of solid } = \lim_{\Delta x \to 0} \sum \pi(1 - x^4)\, \Delta x = \int_0^1 \pi(1 - x^4)\, dx = \pi \left(x - \frac{x^5}{5} \right)\bigg|_0^1 = \frac{4}{5}\pi.$$

28. Find the volume obtained when the region bounded by $y = e^x$, the x-axis, and the lines $x = 0$ and $x = 1$ is rotated around the x-axis.

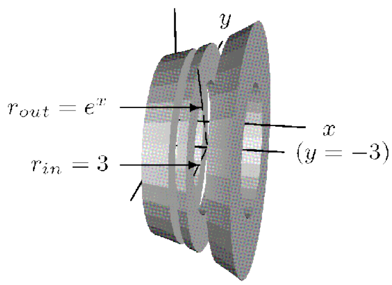


This is the volume of revolution gotten from the rotating the curve $y = e^x$. Take slices perpendicular to the x-axis. They will be circles with radius e^x, so

$$V = \int_{x=0}^{x=1} \pi y^2\, dx = \pi \int_0^1 e^{2x}\, dx$$

$$= \frac{\pi e^{2x}}{2}\bigg|_0^1 = \frac{\pi(e^2 - 1)}{2} \approx 10.036.$$

29. Find the volume obtained when the region bounded by $y = e^x$, the x-axis, and the lines $x = 0$ and $x = 1$ is rotated around the line $y = -3$.



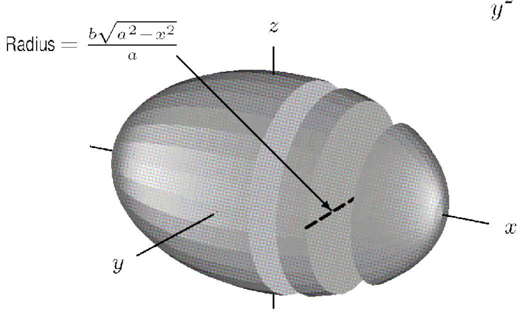
We slice the volume with planes perpendicular to the line $y = -3$. This divides the curve into thin washers, as in Example 3 on page 400 of the text, whose volumes are

$$\pi r_{\text{out}}^2 dx - \pi r_{\text{in}}^2 dx = \pi(3+y)^2 dx - \pi 3^2 dx.$$

So the integral we get from adding all these washers up is

$$\begin{aligned} V &= \int_{x=0}^{x=1} [\pi(3+y)^2 - \pi 3^2]\, dx \\ &= \pi \int_0^1 [(3+e^x)^2 - 9]\, dx \\ &= \pi \int_0^1 [e^{2x} + 6e^x]\, dx = \pi[\frac{e^{2x}}{2} + 6e^x]\Big|_0^1 \\ &= \pi[(e^2/2 + 6e) - (1/2 + 6)] \approx 42.42. \end{aligned}$$

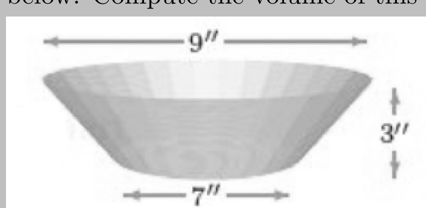
30. Rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the x-axis generates an ellipsoid. Compute its volume.



$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right).$$

$$\begin{aligned} V &= \int_{-a}^{a} \pi y^2 dx = \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx \\ &= 2\pi b^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi b^2 \left[x - \frac{x^3}{3a^2}\right]_0^a \\ &= 2\pi b^2 \left(a - \frac{a^3}{3a^2}\right) = 2\pi b^2 \left(a - \frac{1}{3}a\right) \\ &= \frac{4}{3}\pi a b^2. \end{aligned}$$

31. (a) A pie dish is 9 inches across the top, 7 inches across the bottom, and 3 inches deep, as shown in the figure below. Compute the volume of this dish.



(b) Make a rough estimate of the volume in cubic inches of a single cut-up apple, and estimate the number of apples that is needed to make an apple pie that fills this dish.

(a) We can begin by slicing the pie into horizontal slabs of thickness Δh located at height h. To find the radius of each slice, we note that radius increases linearly with height. Since $r = 4.5$ when $h = 3$ and $r = 3.5$ when $h = 0$, we should have $r = 3.5 + h/3$. Then the volume of each slab will be $\pi r^2\, \Delta h = \pi(3.5 + h/3)^2\, \Delta h$. To find the total volume of the pie, we integrate this from $h = 0$ to $h = 3$:

$$\begin{aligned} V &= \pi \int_0^3 \left(3.5 + \frac{h}{3}\right)^2\, dh \\ &= \pi \left[\frac{h^3}{27} + \frac{7h^2}{6} + \frac{49h}{4}\right]\Bigg|_0^3 \\ &= \pi \left[\frac{3^3}{27} + \frac{7(3^2)}{6} + \frac{49(3)}{4}\right] \approx 152 \text{ in}^3. \end{aligned}$$

(b) We use 1.5 in as a rough estimate of the radius of an apple. This gives us a volume of $(4/3)\pi(1.5)^3 \approx 10 \text{ in}^3$. Since $152/10 \approx 15$, we would need about 15 apples to make a pie.