Unit #18 - Level Curves, Partial Derivatives

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Contour Diagrams

1. Figure 1 shows the density of the fox population P (in foxes per square kilometer) for southern England.

   Draw two different cross-sections along a north-south line and two different cross-sections along an east-west line of the population density P.

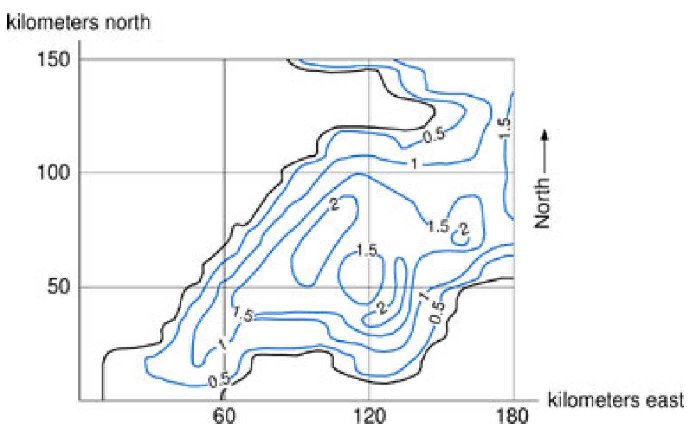


Figure 1

In Problems 2-6, sketch a contour diagram for the function with at least four labeled contours. Describe in words the contours and how they are spaced.

2. $f(x, y) = x + y$

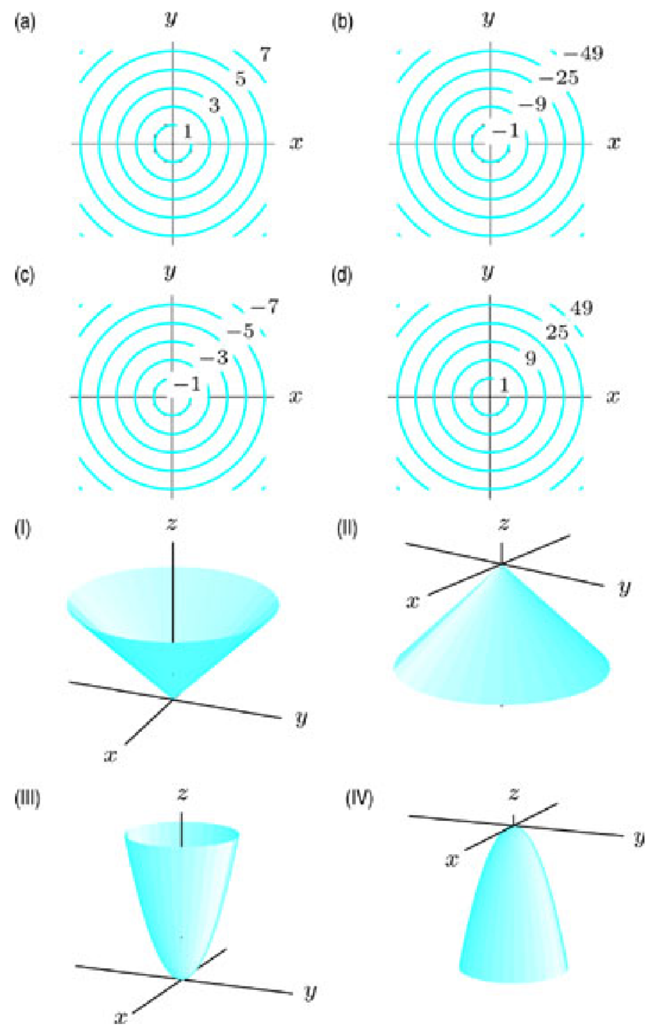
3. $f(x, y) = 3x + 3y$

4. $f(x, y) = x^2 + y^2$

5. $f(x, y) = -x^2 - y^2 + 1$

6. $f(x, y) = \cos\sqrt{x^2 + y^2}$

7. Match the contour diagrams (a)-(d) with the surfaces (I)-(IV). Give reasons for your choice.



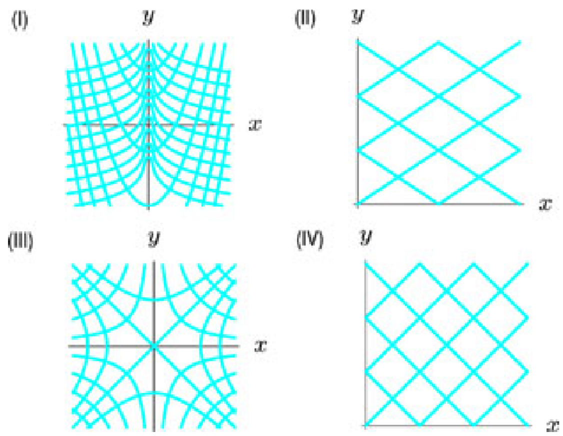
8. Match the pairs of functions (a)-(d) with the contour diagrams (I)-(IV). In each case, show which contours represent f and which represent g. (The x- and y-scales are equal.)

   (a) $f(x, y) = x + y$, $g(x, y) = x - y$

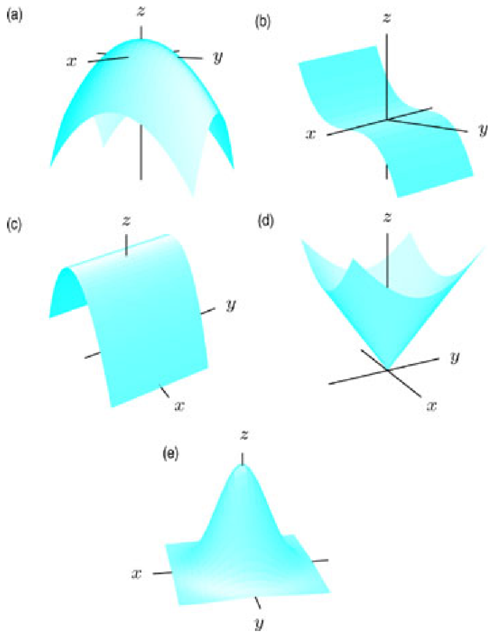
   (b) $f(x, y) = 2x + 3y$, $g(x, y) = 2x - 3y$

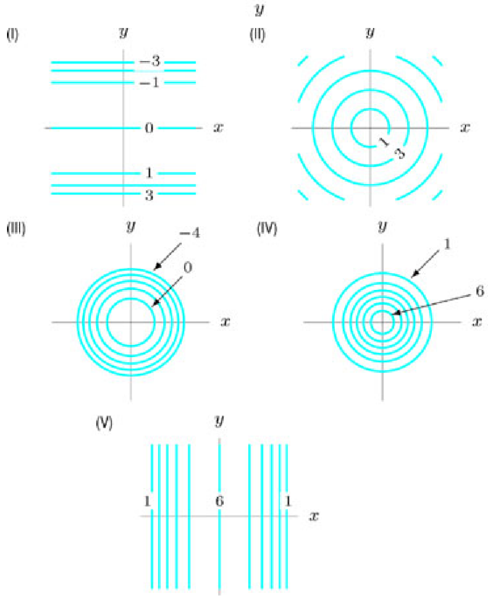
   (c) $f(x, y) = x^2 - y$, $g(x, y) = 2y + \ln|x|$

   (d) $f(x, y) = x^2 - y^2$, $g(x, y) = xy$



9. Match the surfaces (a)-(e) with the contour diagrams (I)-(V) below.





10. Match Tables A-D with the contour diagrams (I)-(IV).

Table A

		x		
		-1	0	1
	-1	2	1	2
y	0	1	0	1
	1	2	1	2

Table B

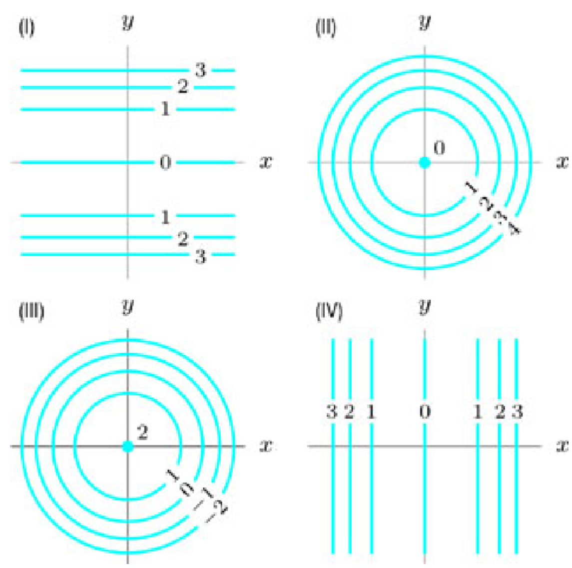
		x		
		-1	0	1
	-1	0	1	0
y	0	1	2	1
	1	0	1	0

Table C

		x		
		-1	0	1
	-1	2	0	2
y	0	2	0	2
	1	2	0	2

Table D

		x		
		-1	0	1
	-1	2	2	2
y	0	0	0	0
	1	2	2	2



11. Match each Cobb-Douglas production function (a)-(c) with a graph (I)-(III) and a statement (D)-(G).

a. $F(L,K) = L^{0.25}K^{0.25}$

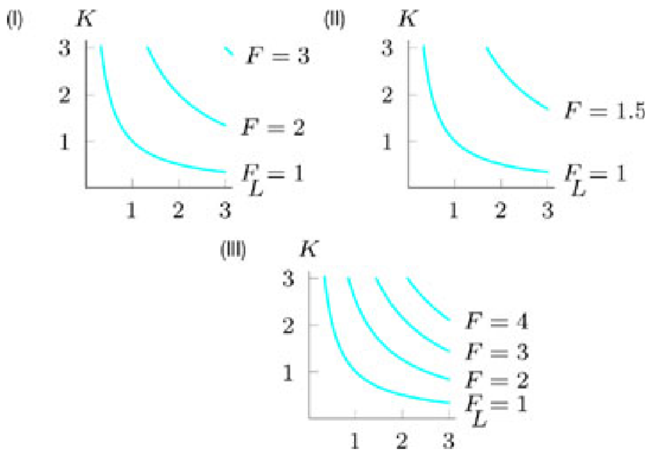
b. $F(L,K) = L^{0.5}K^{0.5}$

c. $F(L,K) = L^{0.75}K^{0.75}$

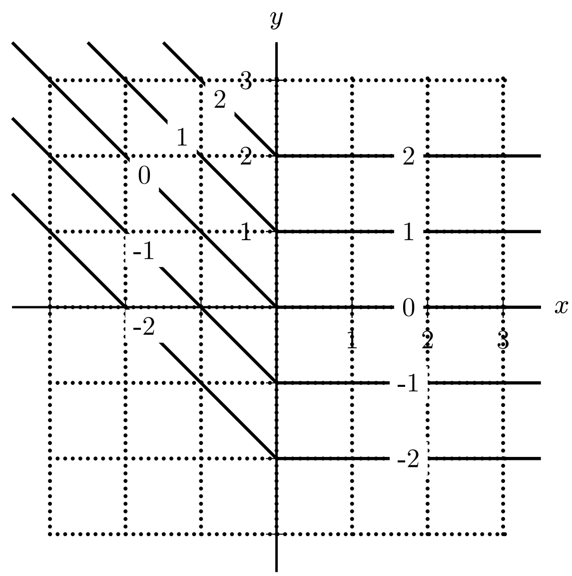
D. Tripling each input triples output.

E. Quadrupling each input doubles output.

G. Doubling each input almost triples output.



12. Below is the contour diagram of $f(x, y)$.



Sketch the contour diagram of each of the following functions.

(a) $3f(x, y)$

(b) $f(x, y) - 10$

(c) $f(x - 2, y - 2)$

(d) $f(-x, y)$

Linear Functions

13. The charge, C, in dollars, to use an Internet service is a function of m, the number of months of use, and t, the total number of minutes on-line:

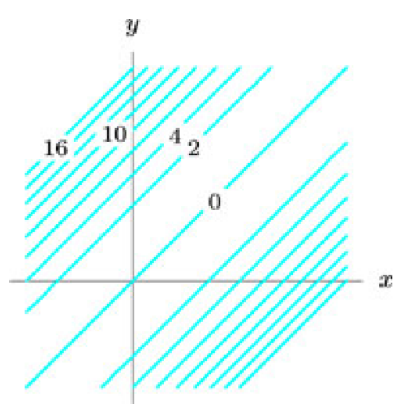
$$C = f(m, t) = 35 + 15m + 0.05t$$

(a) Is f a linear function?

(b) Give units for the coefficients of m and t, and interpret them as charges.

(c) Interpret the intercept 35 as a charge.

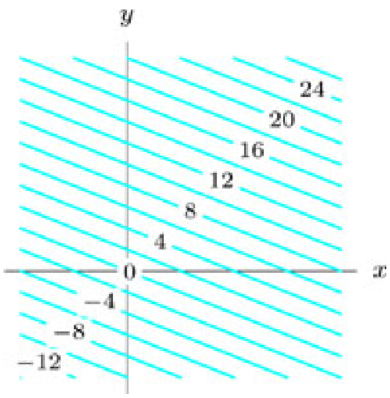
(d) Find $f(3, 800)$ and interpret your answer.



Which of the contour diagrams in Problems **??**-**??** could represent linear functions?

14.

15.



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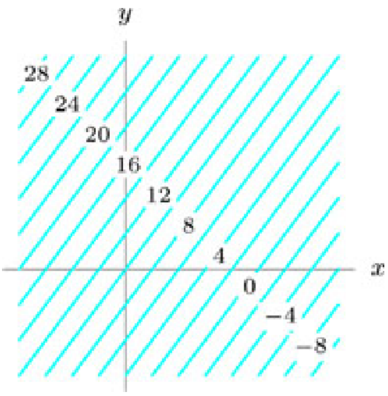

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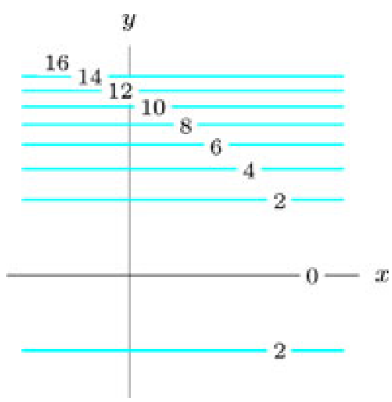


18. Each column of the table below is linear with the same slope, $m = \frac{\Delta z}{\Delta x} = 4/5$. Each row is linear with the same slope,$n = \frac{\Delta z}{\Delta y} = 3/2$. We now investigate the slope obtained by moving through the table along lines that are neither rows nor columns.

		y				
		4	6	8	10	12
	5	3	6	9	12	15
	10	7	10	13	16	19
x	15	11	14	17	20	23
	20	15	18	21	24	27
	25	19	22	25	28	31

(a) Move down the diagonal of the table from the upper left corner ($z = 3$) to the lower right corner ($z = 31$). What do you notice about the changes in z? Now move diagonally from $z = 6$ to $z = 27$. What do you notice about the changes in z now?

(b) Move in the table along a line right one step, up two steps, e.g. from $z = 19$ ($x = 25, y = 4$) to $z = 14$ ($x = 15, y = 6$). Then move in the same direction from $z = 22$ to $z = 17$. What do you notice about the changes in z?

(c) Show that $\Delta z = m\Delta x + n\Delta y$. Use this to explain what you observed in parts (a) and (b).

Which of the tables of values in Exercises **??** -**??** could represent linear functions?

19.

		y		
		0	1	2
	0	0	1	4
x	1	1	0	1
	2	4	1	0

20.

		y		
		0	1	2
	0	10	13	16
x	1	6	9	12
	2	2	5	8

21.

		y		
		0	1	2
	0	0	5	10
x	1	2	7	12
	2	4	9	14

22.

		y		
		0	1	2
	0	5	7	9
x	1	6	9	12
	2	7	11	15

23. Find the linear function whose graph is the plane through the points (4, 0, 0), (0, 3, 0) and (0, 0, 2).

24. Find the equation of the linear function $z = c+mx+ny$ whose graph intersects the xz-plane in the line $z = 3x+4$ and intersects the yz-plane in the line $z = y+4$.

25. Find the equation for the linear function described by the table below.

		y			
		10	20	30	40
	100	3	6	9	12
x	200	2	5	8	11
	300	1	4	7	10
	400	0	3	6	9

The Partial Derivative

26. A drug is injected into a patient's blood vessel. The function $c = f(x, t)$ represents the concentration of the drug (in mg/L) at a distance x mm in the direction of the blood flow measured from the point of injection and at time t seconds since the injection.

    For the following partial derivatives,

    - What are the units of the following partial derivatives?
    - What are their practical interpretations?
    - What do you expect their signs to be?

    (a) $\dfrac{\partial c}{\partial x}$

    (b) $\dfrac{\partial c}{\partial t}$

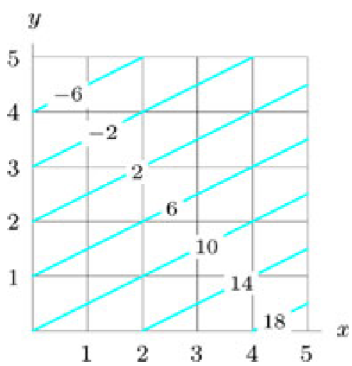
27. The quantity, Q, of beef purchased at a store, in kilograms per week, is a function of the price of beef, b, and the price of chicken, c, both in dollars per kilogram.

    (a) Do you expect $\dfrac{\partial Q}{\partial b}$ to be positive or negative? Explain.

    (b) Do you expect $\dfrac{\partial Q}{\partial c}$ to be positive or negative? Explain.

    (c) Interpret the statement $\dfrac{\partial Q}{\partial b} = -213$ in terms of quantity of beef purchased.

28. Below is a contour diagram for $z = f(x, y)$. Is f_x positive or negative? Is f_y positive or negative? Estimate $f(2, 1)$, $f_x(2, 1)$, and $f_y(2, 1)$.



29. An experiment to measure the toxicity of formaldehyde yielded the data in the table below. The values show the percent, $P = f(t, c)$, of rats surviving an exposure to formaldehyde at a concentration of c (in parts per million, ppm) after t months.

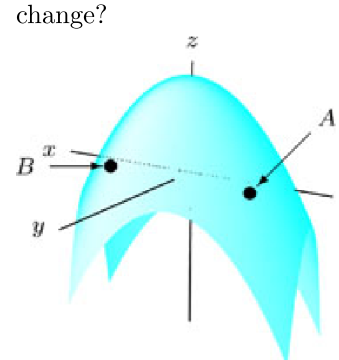
    Estimate $f_t(18, 6)$ and $f_c(18, 6)$. Interpret your answers in terms of formaldehyde toxicity.

		Time t (months)					
		14	16	18	20	22	24
	0	100	100	100	99	97	95
Conc. c	2	100	99	98	97	95	92
(ppm)	6	96	95	93	90	86	80
	15	96	93	82	70	58	36

30. The surface $z = f(x, y)$ is shown in the graph below. The points A and B are in the xy-plane.

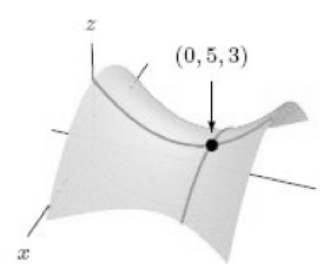
    (a) What is the sign of
    (i) $f_x(A)$?
    (ii) $f_y(A)$?

    (b) The point P in the xy-plane moves along a straight line from A to B. How does the sign of $f_x(P)$ change? How does the sign of $f_y(P)$ change?



NOTE: the axes are not positioned in the usual location! Positive x values are back and left, and positive y values are down and left. This affects your interpretation of the slope.

31. Consider the graph below:

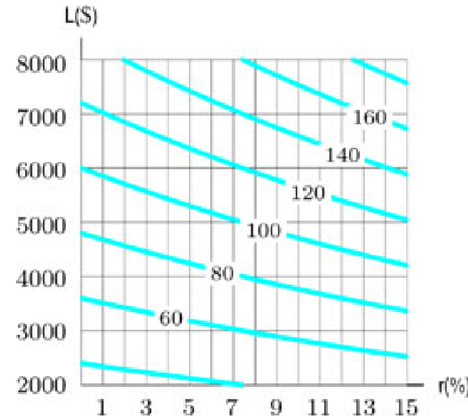


    (a) What is the sign of $f_x(0,\ 5)$?

    (b) What is the sign of $f_y(0,\ 5)$?

32. The figure below shows a contour diagram for the monthly payment P as a function of the interest rate, r %, and the amount, L, of a 5-year loan. Estimate $\dfrac{\partial P}{\partial r}$

and $\dfrac{\partial P}{\partial L}$ at the following points. In each case, give the units and the everyday meaning of your answer.



(a) $r = 8,\ L = 4000$

(b) $r = 8,\ L = 6000$

(c) $r = 13,\ L = 7000$

Computing Partial Derivatives

Find the partial derivatives in Problems **??**–**??**. Assume the variables are restricted to a domain on which the function is defined.

33. f_x and f_y if $f(x, y) = 5x^2y^3 + 8xy^2 - 3x^2$

34. $\dfrac{\partial}{\partial x}(a\sqrt{x})$

35. $\dfrac{\partial}{\partial B}\left(\dfrac{1}{u_0}B^2\right)$

36. F_v if $F = \dfrac{mv^2}{r}$

37. $\dfrac{\partial T}{\partial l}$ if $T = 2\pi\sqrt{\dfrac{l}{g}}$

38. f_a if $f(a, b) = e^a \sin(a + b)$

39. g_x if $g(x, y) = \ln(ye^{xy})$

40. $\dfrac{\partial Q}{\partial K}$ if $Q = c(a_1K^{b_1} + a_2L^{b_2})^{\gamma}$

41. Money in a bank account earns interest at a continuous rate, r. The amount of money, $\$B$, in the account depends on the amount deposited, $\$P$, and the time, t, it has been in the bank according to the formula

$B = Pe^{rt}$

Find $\dfrac{\partial B}{\partial t}$ and $\dfrac{\partial B}{\partial P}$ and interpret each in financial terms.

42. The Dubois formula relates a person's surface area, s, in m^2, to weight, w, in kg, and height, h, in cm, by

$s = f(w, h) = 0.01w^{0.25}h^{0.75}$

Find $f(65, 160)$, $f_w(65, 160)$, and $f_h(65, 160)$. Interpret your answers in terms of surface area, height, and weight.

43. Is there a function f which has the following partial derivatives? If so what is it? Are there any others?

$f_x(x, y) = 4x^3y^2 - 3y^4$

$f_y(x, y) = 2x^4y - 12xy^3$