Unit #22 - Unconstrained Optimization

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Local Extrema

1. Which of the points \( A, B, C \) in the contour diagram below appear to be critical points? Classify those that are critical points.

- A is not a critical point.
- B is (or is very near) a critical point. From the contours nearby, which are increasing towards 2, it is a local maximum.
- C is a critical point. It is a saddle point.

To determine the type of critical point, we need the second derivative information.

\[
\begin{align*}
f_{xx} &= -6, f_{yy} = -10, f_{xy} = 2 \\
D &= f_{xx}f_{yy} - (f_{xy})^2 = 56 > 0
\end{align*}
\]

Since \( D > 0 \) and \( f_{xx} < 0 \), the critical point at (1,5) must be a local maximum.

For Problems 2-5, find the critical points and classify them as local maxima, local minima, saddle points, or none of these.

2. \( f(x, y) = 400 - 3x^2 - 4x + 2xy - 5y^2 + 48y \)

\[
\begin{align*}
f_x &= -6x - 4 + 2y \\
f_y &= 2x - 10y + 48
\end{align*}
\]

If \( f_x = 0 \) : \( 0 = -6x - 4 + 2y \)
\[
y = 2 + 3x
\]

If \( f_y = 0 \) : \( 0 = 2x - 10y + 48 \)
\[
y = \frac{1}{5}(x + 24)
\]

For both derivatives to be zero simultaneously, we must have
\[
2 + 3x = \frac{1}{5}(x + 24) \\
10 + 15x = x + 24
\]
\[
x = 1
\]

Subbing back into an earlier equation, \( y = 5 \)

Thus, there is a single critical point at \((x, y) = (1, 5)\).

3. \( f(x, y) = x^3 - 3x + y^3 - 3y \)

\[
\begin{align*}
f_x &= 3x^2 - 3 \\
f_y &= 3y^2 - 3
\end{align*}
\]

If \( f_x = 0 \) : \( x^2 = 1 \)
\[
x = \pm 1
\]

If \( f_y = 0 \) : \( y^2 = 1 \)
\[
y = \pm 1
\]

We will have critical points at four locations: (1, 1), (-1, 1), (1, -1), and (-1, -1).

To determine the type of each critical point, we need the second derivative information.

\[
\begin{align*}
f_{xx} &= 6x, f_{yy} = 6y, f_{xy} = 0
\end{align*}
\]

At (1,1), \( D = 36, f_{xx} > 0 \) \Rightarrow local minimum

At (-1,1), \( D = -36, \) \Rightarrow saddle point

At (1,-1), \( D = -36, \) \Rightarrow saddle point

At (-1,-1), \( D = 36, f_{xx} < 0 \) \Rightarrow local maximum

4. \( f(x, y) = x^3 + y^3 - 6y^2 - 3x + 9 \)

\[
\begin{align*}
f_x &= 3x^2 - 3 \\
f_y &= 3y^2 - 12y
\end{align*}
\]

If \( f_x = 0 \) : \( x^2 = 1 \)
\[
x = \pm 1
\]

If \( f_y = 0 \) : \( 0 = 3y(y - 4) \)
\[
y = 0, 4
\]
We will have a critical point at four locations: (1, 0), (-1, 0), (1, 4), or (-1, 4).

To determine the type of each critical point, we need the second derivative information.

\[ f_{xx} = 6x, f_{yy} = 6y - 12, f_{xy} = 0 \]

At (1,0), \( D = -72 \Rightarrow \) saddle point
At (-1,0), \( D = 72, f_{xx} < 0 \Rightarrow \) local maximum
At (1,4), \( D = (6)(12) > 0, f_{xx} > 0 \Rightarrow \) local minimum
At (-1,4), \( D = (-6)(12) \Rightarrow \) saddle point

5. \( f(x, y) = 8xy - \frac{1}{4}(x + y)^4 \)

\[ f_x = 8y - (x + y)^3 \]
\[ f_y = 8x - (x + y)^3 \]

Setting both equations to zero gives

\[ 8y - (x + y)^3 = 0 \quad (1) \]
\[ 8x - (x + y)^3 = 0 \quad (2) \]

We can solve for \((x + y)^3\) in both:

\[ (x + y)^3 = 8y \]
\[ (x + y)^3 = 8x \]
so \(8y = 8x\) or \(x = y\) \( (3) \)

Subbing (3) into (1) gives

\[ 8y - (2y)^3 = 0 \]
\[ 8y(1 - y^2) = 0 \]
So \(y = 0\) or \(y = \pm 1\) \( (4) \)

Combining (3) and (4) the critical points are \((x, y) = (-1, -1), (0, 0)\) and \((1, 1)\)

To determine the type of each critical point, we need the second derivative information.

\[ f_{xx} = -3(x+y)^2, f_{yy} = -3(x+y)^2, f_{xy} = 8 - 3(x+y)^2 \]

At (-1,-1), \( D = (-12)(-12) - (8 - 12)^2 = 128 > 0, \)
\( f_{xx} = -12 < 0 \Rightarrow \) local maximum
At (0,0), \( D = -64 \Rightarrow \) saddle point
At (1,1), \( D = (-12)(-12) - (8 - 12)^2 = 128 > 0, \)
\( f_{xx} = -12 < 0 \Rightarrow \) local maximum

6. Find \( A \) and \( B \) so that \( f(x, y) = x^2 + Ax + y^2 + B \) has a local minimum value of 20 at (1, 0).

To have a local minimum at (1, 0), the function must have a critical point there. i.e. both \( f_x \) and \( f_y \) are 0 at the point (1, 0).

\[ f_x = 2x + A \quad \text{so at (1,0), } 0 = 2(1) + A \]
\[ f_y = 2y \quad \text{so at (1,0), } 0 = 2(0) \]

The first equation tells us that \( A \) must equal -2. The second equation verifies that the \( y \) derivative will be zero at (1,0) regardless of the values we pick for \( A \) and \( B \).

The second piece of information we have is that the function has a value of 20 at (1,0):

\[ 20 = f(1,0) \]
\[ 20 = (1)^2 + (-2)(1) + 0^2 + B \]
\[ B = 21 \]

Note: the statement that we are looking for values of \( A \) and \( B \) that make the (1,0) specifically a local minimum (instead of just a critical point) is a bit of a red herring: the values of \( A \) and \( B \) will have no effect on the type of critical point because of where they occur in the function (linear and constant terms that wouldn’t be present in the second derivatives, which determine the type of critical point).

7. Let \( f(x, y) = kx^2 + y^2 - 4xy \)

Determine the values of \( k \) (if any) for which the critical point at (0,0) is:

(a) A saddle point
(b) A local maximum
(c) A local minimum

We are told in advance that there is a critical point at (0,0). Since we need to find the derivatives of \( f \) for the second derivatives test though, we might as well verify that along the way.

\[ f_x = 2kx - 4y \quad \text{at (x, y) = (0,0), } f_x =0 \]
\[ f_y = 2y - 4x \quad \text{at (x, y) = (0,0), } f_y =0 \]

To evaluate what type of critical point we have at (0,0), we find the second derivatives of \( f \) and use the second
derivative test.

\[
f_{xx} = 2k \\
f_{yy} = 2 \\
f_{xy} = -4 \\
\]

so \( D = (2k)(2) - (-4)^2 = 4k - 16 \)

We note that \( D \) will be positive if \( k > 4 \). If \( k > 4 \) then \( f_{xx} = 2k > 8 \)

- If \( k > 4 \), then \( D > 0 \) and \( f_{xx} > 0 \), so the critical point is a local **minimum**.
- If \( k < 4 \), then \( D < 0 \) and the critical point is **saddle**.
- If \( k = 4 \), then \( D = 0 \) and we cannot determine the type of critical point at \((0,0)\) using the second derivative test.

For Problems 8-10, use the contours of \( f \) in the figure below.

8. Decide whether you think each point is a local maximum, local minimum, saddle point, or none of these.

(a) \( P \)
(b) \( Q \)
(c) \( R \)
(d) \( S \)

(a) \( P \) is a local maximum (middle of higher and higher contours)
(b) \( Q \) is a saddle point (crossing of two contours)
(c) \( R \) is a local minimum. This one is a little tougher to see, but if you follow a path from the right, the contour levels are going down. You can also try to draw a graph of the function along a path along \( y = 0 \), which will show the relative height at \( R \) as a local minimum more clearly.
(d) \( S \) is on a single contour, so it is not a critical point. The gradient at a critical point must be the zero vector, but there is a non-zero gradient at \( S \).

9. Sketch the direction of \( \nabla f \) at several points around each of \( P, Q, \) and \( R \).

The gradient (direction of locally steepest ascent) always points

- uphill, and
- perpendicular to the contour.

Around \( P \) and \( Q \), these directions are straightforward to infer. Around \( R \) though, you need to be a little more careful, looking closely at the adjacent contour heights (-1 and 0).

10. At the points where \( ||\nabla f|| \) is largest, put arrows showing the direction of \( \nabla f \).

The gradient’s length, \( ||\nabla f|| \), is equal to the slope in the direction of the gradient. Clusters of contours closer together means steeper slopes, so there are four points
11. The behavior of a function can be complicated near a critical point where $D = 0$. Suppose that 

$$f(x, y) = x^3 - 3xy^2$$

Show that there is one critical point at $(0, 0)$ and that $D = 0$ there. Show that the contour for $f(x, y) = 0$ consists of three lines intersecting at the origin and that these lines divide the plane into six regions around the origin where $f$ alternates from positive to negative. Sketch a contour diagram for $f$ near $(0, 0)$. The graph of this function is called a monkey saddle

$$f_x = 3x^2 - 3y^2$$
$$f_y = -6xy$$

At $(0, 0)$, $f_x = 0$ and $f_y = 0$, so it is a critical point. Also, $f(0,0) = 0$, so the surface has height $z = 0$ at that point. The contours of $z = 0$ occur when

$$x^3 - 3xy^2 = 0$$
$$x(x^2 - 3y^2) = 0$$
So either $x = 0$
$$or 3y^2 = x^2$$

Taking roots, $y = \pm \frac{1}{\sqrt{3}} x$

Thus, any points along the lines $x = 0$, $y = x/\sqrt{3}$ or $y = -x/\sqrt{3}$ all have level zero. These three contours cross at the origin.

By checking the sign of $f$ in between these lines, you can determine that the sign of $f$ alternates from positive to negative as you move around the origin. A contour map is shown below.

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12. A missile has a guidance device which is sensitive to both temperature, $t^\circ$ C, and humidity, $h$. The range in km over which the missile can be controlled is given by

$$\text{Range} = 27,800 - 5t^2 - 6ht - 3h^2 + 400t + 300h$$

What are the optimal atmospheric conditions for controlling the missile?

If we are looking for the optimal conditions for the range, then we want to find a maximum of the Range function. We can do this by looking for critical points.

$$R_t = -10t - 6h + 400$$
$$R_h = -6t - 6h + 300$$

Setting these derivatives equal to zero to find critical points and simplifying, the obtain the equations

$$0 = -5t - 3h + 200$$
$$0 = -t - h + 50$$

Equation 6 leads to $t = 50 - h$. Substituting that into
Equation 5,
\[ 0 = -5(50 - h) - 3h + 200 \]
\[ 50 = 2h \]
\[ h = 25 \]
which leads to \( t = 50 - h = 25 \) as well.
The second derivative shows us that this is a local maximum for the range:
\[ R_{tt} = -10 \]
\[ R_{hh} = -6 \]
\[ R_{ht} = -6 \]
so \( D = (-10)(-6) - (-6)^2 = 24 \) is positive and \( R_{tt} \) is negative (concave down), showing that our critical point is a local maximum of the range.

13. An open rectangular box has volume 32 cm\(^3\). What are the lengths of the edges giving the minimum surface area? (You may assume that the answer you find is a local minimum for the surface area.)

If we let \( x \) and \( y \) be the length and width of the box, and \( z \) be the height, the total surface area is
\[ S = \text{bottom} + 2 \text{sides} + 2 \text{ends} \]
\[ = xy + 2xz + 2yz \]
Since we need the volume to be 32 cm\(^3\), then we must have \( 32 = xyz \), so we can simplify \( z = \frac{32}{xy} \)

\[ S = xy + 2 \frac{32x}{xy} + 2 \frac{32y}{xy} \]
\[ = xy + \frac{64}{y} + \frac{64}{x} \]
Set \( \frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0 \)
gives \( y = \frac{64}{x^2} \)
and setting \( \frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0 \)
gives \( x = \frac{64}{y^2} \)
Putting both constraints and \( x \) and \( y \) together, we get
\[ y = \frac{64}{x^2} = \frac{64}{\left(\frac{64}{y^2}\right)^2} = \frac{y^4}{64} \]
so \( y = 0 \) or \( y^3 = 64 \)
\[ y = 4 \text{ cm} \]
Knowing that \( y = 4 \) gives \( x = \frac{64}{y^2} = 4 \text{ cm} \) as well (not surprising, given the symmetry of the box). Finally, that means that the height is \( z = \frac{32}{xy} = 2 \text{ cm} \).

It is straightforward to check, using the second derivative test, that this is a local minimum for \( S \). Checking that it is a global minimum is more than we have seen in this course.

14. A closed rectangular box with faces parallel to the coordinate planes has one bottom corner at the origin and the opposite top corner in the first octant on the plane
\[ 3x + 2y + z = 1 \]

What is the maximum volume of such a box? (Verify that your answer is a local maximum for the volume.)

The volume of the box is given by
\[ V = xyz \]
To reduce the problem to 2 variables, we use the fact that \( z \) is constrained by the planes, so
\[ z = 1 - 3x - 2y \]
so
\[ V(x, y) = xy(1 - 3x - 2y) = xy - 3x^2y - 2xy^2 \]
We now look for critical points,
Setting \( \frac{\partial V}{\partial x} = y - 6xy - 2y^2 = 0 \)
gives \( x = \frac{y - 2y^2}{6y} = \frac{1 - 2y}{6} \)
Setting \( \frac{\partial V}{\partial y} = x - 3x^2 - 4xy = 0 \)
gives \( y = \frac{x - 3x^2}{4x} = \frac{1 - 3x}{4} \)
Putting the first and second equations together gives
\[ x = \frac{1 - 2y}{6} = \frac{1 - 2(1 - 3x)/4}{6} = \frac{1 - 2 + 6x}{4} = 24x = 4 - 2 + 6x \]
\[ 18x = 2 \]
\[ x = \frac{1}{5} \]
and then \( y = \frac{1 - 3x}{4} = \frac{1}{6} \)
Finally, this gives us \( z = 1 - 3x - 2y = \frac{1}{3} \). Thus, we should place the corner of the box at \( (x, y, z) = \left( \frac{1}{9}, \frac{1}{6}, \frac{1}{3} \right) \).

To verify that this is a local maximum for the volume, we use the second derivative test:

\[
\begin{align*}
\frac{\partial^2 V}{\partial x^2} &= -6y \\
\frac{\partial^2 V}{\partial y^2} &= -4x \\
\frac{\partial^2 V}{\partial x \partial y} &= 1 - 6x - 4y
\end{align*}
\]

At the point \( (x, y, z) = \left( \frac{1}{9}, \frac{1}{6}, \frac{1}{3} \right) \),

\[
\begin{align*}
\frac{\partial^2 V}{\partial x^2} &= -6y = -1 \\
\frac{\partial^2 V}{\partial y^2} &= -4x = -\frac{4}{9} \\
\frac{\partial^2 V}{\partial x \partial y} &= 1 - 6x - 4y = \frac{1}{3}
\end{align*}
\]

\[D = (-1)(-\frac{4}{9}) - \left( \frac{1}{3} \right)^2 = \frac{1}{3} > 0\]

and since \( V_{xx} < 0 \), the critical point is a local maximum for the volume.

15. Find the point on the plane

\[3x + 2y + z = 1\]

that is closest to the origin by minimizing the square of the distance. Confirm that your answer is a local minimum of the distance.

A point on the plane \( 3x + 2y + z = 1 \) can be defined just by the \( (x, y) \) coordinate, with \( z = 1 - 3x - 2y \) as the third value.

The distance from any point to the origin is

\[\text{dist} = \sqrt{x^2 + y^2 + z^2}\]

so the distance from a point on this plane (with \( z \) defined as shown earlier) is

\[\text{dist} = \sqrt{x^2 + y^2 + (1 - 3x - 2y)^2}\]

We are trying to find the \( (x, y) \) point with the minimum distance value. We could minimize the distance directly, but because of the square root the derivatives would be unnecessarily complicated. Instead, we will follow the tip in the question and minimize the square of the distance instead, which is smallest when the distance is smallest:

\[L(x, y) = \text{dist squared}(x, y) = x^2 + y^2 + (1 - 3x - 2y)^2\]

To find the minimum value, we identify the critical points by finding both partial derivatives and setting them equal to zero.

\[
\begin{align*}
L_x &= 2x + 2(1 - 3x - 2y) \quad (\text{Verifying that this is a local minimum of the distance gives} \quad x = \frac{3}{14} \quad \text{and} \quad y = \frac{1}{7})
\end{align*}
\]

To confirm that this is a local minimum of the distance squared function, we use the second derivative test.

\[
\begin{align*}
L_{xx} &= 2 + 2(-3)(-3) = 20 \\
L_{yy} &= 2 + 2(-2)(-2) = 10 \\
L_{xy} &= 12
\end{align*}
\]

so \( D \), the second derivative test value is

\[D = L_{xx}L_{yy} - (L_{xy})^2 = (20)(10) - (12)^2 = 200 - 144 = 56\]

and since \( L_{xx} > 0 \) our function is concave up at the critical point, meaning the point we found is a local minimum for the distance function.

16. Two products are manufactured in quantities \( q_1 \) and \( q_2 \) and sold at prices of \( p_1 \) and \( p_2 \), respectively. The cost of producing them is given by

\[C = 2q_1^2 + 2q_2^2 + 10\]

(a) Find the maximum profit that can be made, assuming the prices are fixed. (Verify that the critical point you choose is in fact a local maximum.)

(b) Find the rate of change of that maximum profit as \( p_1 \) increases. I.e. for every dollar you increase the price \( p_1 \), what will the effect be on your profit, assuming you are at the maximum for the current prices?

\[\text{Revenue } R = p_1q_1 + p_2q_2\]

\[\text{Profit } = P = R - C = p_1q_1 + p_2q_2 - (2q_1^2 + 2q_2^2 + 10)\]

\[= p_1q_1 + p_2q_2 - 2q_1^2 - 2q_2^2 - 10\]
We take derivatives and look for critical points by setting them equal to zero.

Set \( \frac{\partial P}{\partial q_1} = p_1 - 4q_1 = 0 \)

\( q_1 = \frac{p_1}{4} \)

and \( \frac{\partial P}{\partial q_2} = p_2 - 4q_2 = 0 \)

\( q_2 = \frac{p_2}{4} \)

This gives us a single critical point that depends on the prices:

\( (q_1, q_2) = \left( \frac{p_1}{4}, \frac{p_2}{4} \right) \)

To check that this is a maximum, we use the second derivative test.

\( \frac{\partial^2 P}{\partial q_1^2} = -4 < 0 \)

\( \frac{\partial^2 P}{\partial q_2^2} = -4 \)

\( \frac{\partial^2 P}{\partial q_1 \partial q_2} = 0 \)

So \( D = (-4)(-4) - 0 > 0 \)

Thus, by the second derivative test, \( P \) has a local maximum value at

\( (q_1, q_2) = \left( \frac{p_1}{4}, \frac{p_2}{4} \right) \)

Since \( P \) is quadratic in \( q_1 \) and \( q_2 \), this is a not just a local maximum, but a global maximum.

Finally, we compute the actual profit at those quantities,

\[
P = \frac{p_1^2}{4} + \frac{p_2^2}{4} - \frac{2p_1^2}{16} - \frac{2p_2^2}{16} - 10 = \frac{p_1^2}{8} + \frac{p_2^2}{8} - 10
\]

is the maximum profit.

(b) The rate of change of the maximum profit as \( p_1 \) increases is

\[
\frac{\partial (\text{max } P)}{\partial p_1} = \frac{2p_1}{8} = \frac{p_1}{4}
\]

At the maximum profit production level, every dollar of increase to the price \( p_1 \) will increase profits by 1/4 of a dollar, or 25 cents.