Unit #23 - Lagrange Multipliers

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Lagrange Multipliers

In Problems 1–4, use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint, if such values exist. Make an argument supporting the classification of your minima and maxima.

1. $f(x,y) = x + y$, $x^2 + y^2 = 1$

We use the constraint to build the constraint function, $g(x,y) = x^2 + y^2$. We then take all the partial derivatives which will be needed for the Lagrange multiplier equations:

$$f_x = 1 \qquad g_x = 2x$$
$$f_y = 1 \qquad g_y = 2y$$

Setting up the Lagrange multiplier equations:

$$f_x = \lambda g_x \quad \Rightarrow \quad 1 = \lambda 2x \tag{1}$$
$$f_y = \lambda g_y \quad \Rightarrow \quad 1 = \lambda 2y \tag{2}$$
$$\text{constraint:} \quad \Rightarrow \quad x^2 + y^2 = 1 \tag{3}$$

Taking (1) / (2), (assuming $\lambda \neq 0$)

$$\frac{1}{1} = \frac{\lambda 2x}{\lambda 2y} = \frac{x}{y}$$
$$\text{so } y = x$$

Sub into (3) to find

$$2x^2 = 1 \quad \Rightarrow \quad x = \pm\sqrt{1/2}$$

Combining with $y = x$, we get the solutions $(x,y) = (\sqrt{1/2}, \sqrt{1/2})$ and $(-\sqrt{1/2}, -\sqrt{1/2})$.

Since our constraint is closed and bounded (only points on the circle $x^2 + y^2 = 1$ are allowed), we can simply compare the value of f at these two points to determine the maximum and minimum values of f subject to the constraint.

$$f(\sqrt{1/2}, \sqrt{1/2}) = 2\sqrt{1/2}$$
$$f(-\sqrt{1/2}, -\sqrt{1/2}) = -2\sqrt{1/2}$$

From this, the maximum of f on $x^2 + y^2 = 1$ is at $(\sqrt{1/2}, \sqrt{1/2})$ and the minimum is at $(-\sqrt{1/2}, -\sqrt{1/2})$

2. $f(x,y) = xy$, $4x^2 + y^2 = 8$

$$f_x = y \qquad g_x = 8x$$
$$f_y = x \qquad g_y = 2y$$

Set up the Lagrange multiplier equations:

$$f_x = \lambda g_x \quad \Rightarrow \quad y = \lambda 8x \tag{4}$$
$$f_y = \lambda g_y \quad \Rightarrow \quad x = \lambda 2y \tag{5}$$
$$\text{constraint:} \quad \Rightarrow \quad 4x^2 + y^2 = 8 \tag{6}$$

Taking (4) / (5), (assuming $\lambda \neq 0$)

$$\frac{y}{x} = \frac{\lambda 8x}{\lambda 2y} = \frac{8x}{2y}$$
$$\text{so } y^2 = 4x^2 \text{ or } y = \pm 2x$$

Sub into (6) to find

$$4x^2 + 4x^2 = 8 \quad \Rightarrow \quad x = \pm 1$$

Combining with $y = \pm 2x$, we get the solutions $(x,y) = (1,2), (1,-2), (-1,2)$ and $(-1,-2)$.

Since our constraint is closed and bounded, we can simply compare the value of f at these four points to determine the maximum and minimum values of f subject to the constraint.

$$f(1,2) = 2$$
$$f(1,-2) = -2$$
$$f(-1,2) = -2$$
$$f(-1,-2) = 2$$

From this,

- the maximum of f on the constraint $4x^2 + y^2 = 8$ is at two points, (1, 2) and (-1, -2); the f value there is +2.
- The minimum of f occurs at (1, -2) and (-1, 2); the f value there is -2.

3. $f(x,y) = x^2 + y$, $x^2 - y^2 = 1$

$$f_x = 2x \qquad g_x = 2x$$
$$f_y = 1 \qquad g_y = -2y$$

Set up the Lagrange multiplier equations:

$$\begin{aligned} f_x = \lambda g_x &\quad\Rightarrow\quad 2x = \lambda(2x) && (7)\\ f_y = \lambda g_y &\quad\Rightarrow\quad 1 = \lambda(-2y) && (8)\\ \text{constraint:} &\quad\Rightarrow\quad x^2 - y^2 = 1 && (9) \end{aligned}$$

From (7), we must have $\lambda = 1$ or $x = 0$

- If $\lambda = 1$, then (8) gives $1 = (1)(-2y)$, or $y = \frac{-1}{2}$, and from (9) $x^2 - \left(\frac{-1}{2}\right)^2 = 1$, so $x = \pm\sqrt{1 + \frac{1}{4}} = \pm\sqrt{\frac{5}{4}}$
- If $x = 0$, then (9) gives $0^2 - y^2 = 1$, but this has no solution! In other words, no point with $x = 0$ belongs to the constraint, so we won't get any candidate points from this option.

The solutions to the Lagrange Multiplier equations are therefore $(x, y) = (\sqrt{\frac{5}{4}}, \frac{-1}{2})$, and $(-\sqrt{\frac{5}{4}}, \frac{-1}{2})$.

The associated function values at these points are:

- $f\left(\sqrt{\frac{5}{4}}, \frac{-1}{2}\right) = x^2 + y = \frac{5}{4} + \frac{-1}{2} = \frac{3}{4}$
- $f\left(-\sqrt{\frac{5}{4}}, \frac{-1}{2}\right) = x^2 + y = \frac{5}{4} + \frac{-1}{2} = \frac{3}{4}$

Since the constraint is *not* bounded, it is not as easy to demonstrate that these values are minimums of f on the constraint. However, with a little mathematical insight it can be done in just a few steps:

$f(x, y) = x^2 + y,$

but we are limited to the constraint

$x^2 - y^2 = 1,$ or $x^2 = y^2 + 1$

Substituting this into f, we get

$f(x, y) = (y^2 + 1) + y = y^2 + y + 1$ on the constraint

Completing the square gives

$$f(x, y) = \left(y + \frac{1}{2}\right)^2 + \frac{3}{4}$$

Since squared values are always positive, we can say that

$$f(x, y) = \left(y + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} \text{ on the constraint curve}$$

Therefore, the values we found $\left(f = \frac{3}{4}\right)$ are minimums of f on the constraint.

[On a test or exam, this kind of check would not be expected without some prompting steps.]

4. $f(x, y) = x^2 + 2y^2, x^2 + y^2 \leq 4$

Note that we are dealing with an inequality for the constraint. We can consider any point **in or on** the boundary of a circle with radius 2. To look **on** the boundary, we use Lagrange multipliers. To look at the **interior**, we identify the critical points of $f(x, y)$.

We'll start with the Lagrange multipliers:

$$\begin{aligned} f_x &= 2x & g_x &= 2x\\ f_y &= 4y & g_y &= 2y \end{aligned}$$

Set up the Lagrange multiplier equations:

$$\begin{aligned} f_x = \lambda g_x &\quad\Rightarrow\quad 2x = \lambda 2x && (10)\\ f_y = \lambda g_y &\quad\Rightarrow\quad 4y = \lambda 2y && (11)\\ \text{constraint:} &\quad\Rightarrow\quad x^2 + y^2 = 4 && (12) \end{aligned}$$

From (10), either $x = 0$ or $\lambda = 1$. If $x = 0$, then (12) says $y = \pm 2$. Alternatively, if $\lambda = 1$, then (11) means $y = 0$, so $x = \pm 2$. Our solutions are

$$(x, y) = (0, 2), (0, -2), (2, 0) \text{ and } (-2, 0)$$

At these points,

$$\begin{aligned} f(0, 2) &= 8\\ f(0, -2) &= 8\\ f(2, 0) &= 4\\ f(-2, 0) &= 4 \end{aligned}$$

Before we can say these are global max or mins, we need to look for critical points in the interior of the circle $x^2 + y^2 \leq 4$.

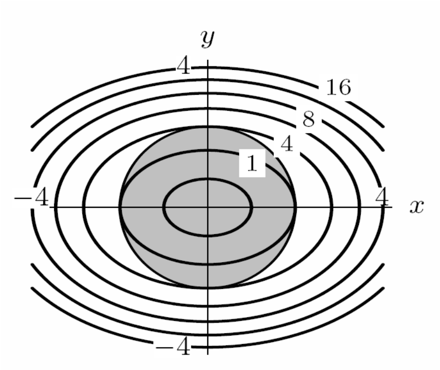
$$\begin{aligned} \text{Set } fx = 0 &\quad\Rightarrow\quad 2x = 0\\ \text{and } f_y = 0 &\quad\Rightarrow\quad 4y = 0 \end{aligned}$$

The only critical points is (0, 0), and this is in the interior of the circle. The value of $f(0, 0) = 0$.

Combining the results on the boundary with the only critical point we see:

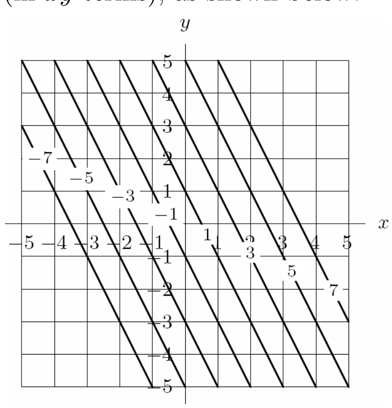
- $f(0, 2)$ and $f(0, -2)$ are global maxes with values of $f = 8$
- $f(0, 0)$ is the global min on the region, with $f = 0$.

A contour diagram showing the region and contours of f is included below to illustrate the solution.

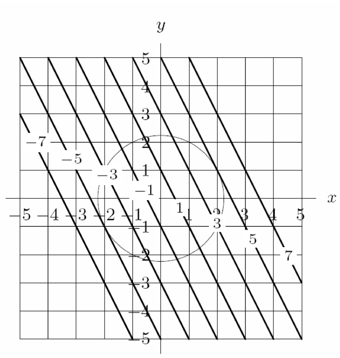


> 5. (a) Draw contours of $f(x,y) = 2x + y$ for $z = -7, -5, -3, -1, 1, 3, 5, 7$.
>   (b) On the same axes, graph the constraint $x^2 + y^2 = 5$.
>   (c) Use the graph to approximate the points at which f has a maximum or a minimum value subject to the constraint $x^2+y^2 = 5$.
>   (d) Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 2x + y$ subject to $x^2 + y^2 = 5$.

(a) The contours of f are straight lines with slope -2 (in xy terms), as shown below.



(b) Overlaying the constraint, we are allowed to move on a circle of radius $\sqrt{5}$.



(c) From the graph, the maximum values occurs where the constraint circle just touches the $f = 5$ contour line, at $(x,y) = (2,1)$. The minimum value is $f = -5$, which occurs on the opposite side of the circle, at $(-2,-1)$.

(d) Computing the constrained optimum locations using Lagrange multipliers,

$$f_x = 2 \qquad g_x = 2x$$
$$f_y = 1 \qquad g_y = 2y$$

Set up the Lagrange multiplier equations:

$$f_x = \lambda g_x \quad \Rightarrow \quad 2 = \lambda 2x \tag{13}$$
$$f_y = \lambda g_y \quad \Rightarrow \quad 1 = \lambda 2y \tag{14}$$
$$\text{constraint:} \quad \Rightarrow \quad x^2 + y^2 = 5 \tag{15}$$

Taking (13) / (14), (assuming $\lambda \neq 0$)

$$\frac{2}{1} = \frac{\lambda 2x}{\lambda 2y} = \frac{x}{y}$$
$$\text{so } 2y = x$$

Sub into (15) to find

$$4y^2 + y^2 = 5 \quad \Rightarrow \quad y = \pm 1$$

Combining with $2y = x$, we get the solutions $(x,y) = (2,1)$ and $(-2,-1)$. These are the same points we found in (c), and knowing their z values, we know that $f(2,1)$ is a maximum while $f(-2,-1)$ is a minimum on the constraint.

> 6. A company manufactures x units of one item and y units of another. The total cost in dollars, C, of producing these two items is approximated by the function
>
> $$C = 5x^2 + 2xy + 3y^2 + 800$$
>
>   (a) If the production quota for the total number of items (both types combined) is 39, find the minimum production cost.
>   (b) Estimate the additional production cost or savings if the production quota is raised to 40 or lowered to 38.

(a) If the total production is 39, then

$$\underbrace{x + y}_{g(x,y)} = \underbrace{39}_{k}$$

This is our constraint in the form $g(x,y) = k$
Setting up the Lagrange multiplier equations,

$$\underbrace{10x + 2y}_{C_x} = \lambda \underbrace{1}_{g_x} \tag{16}$$
$$\underbrace{2x + 6y}_{C_y} = \lambda \underbrace{1}_{g_y} \tag{17}$$
$$x + y = 39 \tag{18}$$

Setting (16) equal to (17),

$$\begin{aligned} 10x + 2y &= 2x + 6y \\ 8x &= 4y \\ y &= 2x \end{aligned}$$

Sub that into (18),

$$\begin{aligned} x + (2x) &= 39 \\ x &= 13 \\ \text{and so } y = 2x &= 26 \end{aligned}$$

The optimal production levels are $x = 13$ units and $y = 26$ units, giving a total production of 39 units.

(b) We are asked to evaluate the impact on the cost of adding one or removing one item from the quota. The Lagrange multiplier value gives us the approximate effect on the cost of adding one unit to the constraint value k, which in this case *is* the change in the quota. Using $x = 12$ and $y = 26$, (16) gives us

$$\lambda = 10(13) + 2(26) = 182$$

so adding one unit to the total production (or producing 40 units) will increase the cost by \$182. Similarly, by removing one unit from the quota (or producing 38 units), the production cost will drop by \$182.

7. A firm manufactures a commodity at two different factories. The total cost of manufacturing depends on the quantities, q_1 and q_2, supplied by each factory, and is expressed by the joint cost function,

$$C = f(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500$$

The company's objective is to produce 200 units, while minimizing production costs. How many units should be supplied by each factory?

We want to minimize

$$C = f(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500$$

subject to the constraint $q_1 + q_2 = 200$ (so $g(q_1, q_2) = q_1 + q_2$).

Since $\nabla f = (4q_1 + q_2, 2q_2 + q_1)$ and $\nabla g = (1, 1)$, setting $\nabla f = \nabla g$ gives

$$\begin{aligned} 4q_1 + q_2 &= \lambda 1 \\ q_1 + 2q_2 &= \lambda 1 \end{aligned}$$

Solving, we get

$$4q_1 + q_2 = q_1 + 2q_2$$

so

$$3q_1 = q_2.$$

We want

$$q_1 + q_2 = 200$$

$$q_1 + 3q_1 = 4q_1 = 200$$

Therefore,

$$q_1 = 50$$

and

$$q_2 = 150$$

From the problem statement, we can conclude that this production level will minimize the total manufacturing cost, given the desired size of production run.

8. Each person tries to balance his or her time between leisure and work. The trade-off is that as you work less your income falls. Therefore each person has indifference curves which connect the number of hours of leisure, l, and income, s. If, for example, you are indifferent between 0 hours of leisure and an income of \$1125 a week on the one hand, and 10 hours of leisure and an income of \$750 a week on the other hand, then the points $l = 0$, $s = 1125$, and $l = 10$, $s = 750$ both lie on the same indifference curve. The table below gives information on three indifference curves, I, II, and III.

Weekly income			Weekly leisure hours		
I	II	III	I	II	III
1125	1250	1375	0	20	40
750	875	1000	10	30	50
500	625	750	20	40	60
375	500	625	30	50	70
250	375	500	50	70	90

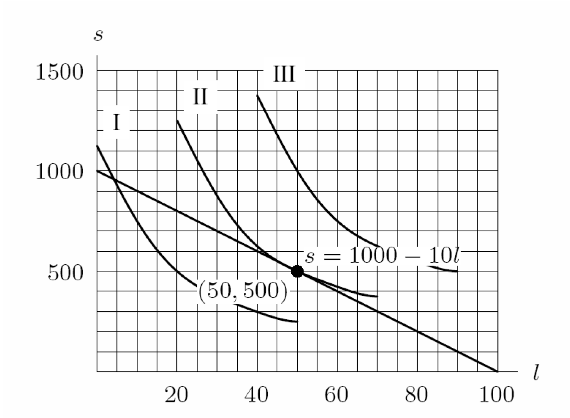
(a) Graph the three indifference curves.

(b) You have 100 hours a week available for work and leisure combined, and you earn \$10/ hour. Write an equation in terms of l and s which represents this constraint.

(c) On the same axes, graph this constraint.

(d) Estimate from the graph what combination of leisure hours and income you would choose under these circumstances. Give the corresponding number of hours per week you would work.

(a) The graphs are shown, along with the constraint from part (c), below.



(b) Since you're earning \$10 per hour, and s is your income, $s/10$ is the number of hours worked. To limit yourself to 100 hours per week, you must satisfy $l + s/10 = 100$

(c) See the graph from (a)

(d) Since the constraint line just touches the indifference curve II at $t = 50, s = 500$, we can't achieve a higher level of satisfaction than level II. To achieve that level of satisfaction, we should split our time into $t = 50$ hours of leisure, and $s/10 = 500/10 = 50$ hours of work.

9. The director of a neighborhood health clinic has an annual budget of \$600,000. He wants to allocate his budget so as to maximize the number of patient visits, V, which is given as a function of the number of doctors, D, and the number of nurses, N, by

$$V = 1000D^{0.6}N^{0.3}$$

A doctor's salary is \$40,000; nurses get \$10,000.

(a) Set up the director's constrained optimization problem.

(b) Describe, in words, the conditions which must be satisfied by $\dfrac{\partial V}{\partial D}$ and $\dfrac{\partial V}{\partial N}$ for V to have an optimum value.

(c) Solve the problem formulated in part (a)

(d) Find the value of the Lagrange multiplier and interpret its meaning in this problem.

(e) At the optimum point, what is the marginal cost of a patient visit (that is, the cost of an additional visit)?

(a) The problem is to maximize

$$V = 1000D^{0.6}N^{0.3}$$

subject to the budget constraint that

$$40,000\ D + 10,000\ N \leq 600,000$$

This will be easier to deal with if we divide by 10,000, so

$$4D + N \leq 60$$

Since there our function, V, always grows larger with larger N and D, there is no point in using less than our budget, so we want to find the maximum value of V on the line $4D + N = 60$. In this problem, then, our constraint function is $g(D, N) = 4D + N$.

(b) We want ∇V to point in the same direction as ∇g, or mathematically that

$$V_D = \lambda g_D$$
$$V_N = \lambda g_N$$

while satisfying the constraint $4D + N = 60$

(c)

$$V_D = 1000(0.6)D^{-0.4}N^{0.3} \qquad g_D = 4$$
$$V_N = 1000(0.3)D^{0.6}N^{-0.7} \qquad g_N = 1$$

Set up the Lagrange multiplier equations:

$$V_D = \lambda g_D \quad \Rightarrow \quad 1000(0.6)D^{-0.4}N^{0.3} = \lambda 4 \qquad (19)$$

$$V_N = \lambda g_N \quad \Rightarrow \quad 1000(0.3)D^{0.6}N^{-0.7} = \lambda 1 \qquad (20)$$

$$\text{constraint:} \quad \Rightarrow \quad 4D + N = 60 \qquad (21)$$

Taking (19) / (20), (assuming $\lambda \neq 0$)

$$\frac{1000(0.6)D^{-0.4}N^{0.3}}{1000(0.3)D^{0.6}N^{-0.7}} = \frac{4}{1}$$
$$2D^{-1}N^{1} = 4$$
$$N = 2D$$

Sub into (21) to find

$$4D + 2D = 60 \quad \Rightarrow \quad D = 10$$

Combining with $N = 2D$, we get the solution that Nurses (N)= 20 and Doctors (D) = 10. This results in $V = 1000(10^{0.6})(20^{0.3}) \approx 9,779$ visits per year.

(d) From (c), we can find λ using (2):

$$\lambda = 1000(0.3)(10)^{0.6}(20)^{-0.7} \approx 146$$

Since $\lambda = \dfrac{\nabla V}{\nabla g}$, it represents the change of visitors for each change in budget. Since our units for the budget constraint were \$10,000 (remember, we divided by 10,000 to simplify in (c)), this means that increasing the budget by one unit (of \$10,000) will result in handling ≈ 146 more visits. More generally, we can handle approximately 0.0146 more patients per dollar increase in budget.

(e) The marginal cost (dollars per patient) is the inverse of the quantity in (d) (patients per dollar). Thus each new patient costs roughly $1/(0.0146$ patients/dollar$) \approx \$68.5$ per patient.

10. A mountain climber at the summit of a mountain wants to descend to a lower altitude as fast as possible. The altitude of the mountain is given approximately by

$$h(x,y) = 3000 - \frac{1}{10,000}(5x^2+4xy+2y^2) \text{ meters}$$

where x, y are horizontal coordinates on the earth (in meters), with the mountain summit located above the origin. In thirty minutes, the climber can reach any point (x,y) on or within a circle of radius 1000 m. What point should she travel to in order to get as far down as possible in 30 minutes?

The mountain climber can reach anywhere in the circle $x^2+y^2 \leq 1000^2$ in the half hour. We want to find the minimum value of h on (or inside) that circle.

To identify any local minima inside the boundary, we first look for critical points:

$$h_x = \frac{-(10x+4y)}{10,000} \qquad h_y = \frac{-(4y+4x)}{10,000}$$

Setting both equal to zero,

$$\frac{-(10x+4y)}{10,000} = 0 \qquad \frac{-(4y+4x)}{10,000} = 0$$

$$5x = -2y \qquad y = -x$$

The only solution to these equations is $x=0, y=0$. This we were already told is the local (and global) maximum at the origin, so we can ignore this in our solution.

Since there are no local minima within the circle the hiker can reach, we now look on the boundary of their 1,000 m reachable circle to determine the lowest she can travel, using Lagrange multipliers. The constraint is $x^2+y^2=1000^2$, or $g(x,y)=x^2+y^2$:

$$h_x = \frac{-(10x+4y)}{10,000} \qquad g_x = 2x$$

$$h_y = \frac{-(4y+4x)}{10,000} \qquad g_y = 2y$$

Set up the Lagrange multiplier equations:

$$h_x = \lambda g_x \quad \Rightarrow \quad \frac{-(10x+4y)}{10,000} = \lambda 2x \qquad (22)$$

$$h_y = \lambda g_y \quad \Rightarrow \quad \frac{-(4y+4x)}{10,000} = \lambda 2y \qquad (23)$$

$$\text{constraint:} \quad \Rightarrow \quad x^2+y^2 = 1000^2 \qquad (24)$$

Taking (22) / (23), (assuming $\lambda \neq 0$)

$$\frac{10x+4y}{4y+4x} = \frac{\lambda 2x}{\lambda 2y} = \frac{x}{y}$$

$$\text{so } (10x+4y)y = x(4y+4x)$$

$$5xy+2y^2 = 2xy+2x^2$$

$$2y^2+3xy-2x^2 = 0$$

$$\text{Factoring: } (2y-x)(y+2x) = 0$$

So either $2y=x$ or $y=-2x$. Using the constraint equation, that gives the possibilities

$$2y = x \quad \Rightarrow \quad 4y^2+y^2 = 1000^2$$

$$y \approx \pm 447, x \approx \pm 894 \ (x,\ y \text{ with same signs})$$

$$y = -2x \quad \Rightarrow \quad x^2+4x^2 = 1000^2$$

$$x \approx \pm 447, y \approx \pm 894 \ (x,\ y \text{ with opposite signs})$$

Substituting these values into the height function, $h(x,y)$, we find that the points with the lowest h values are $(894,447)$ and $(-894,447)$, giving a height of around 2,400 meters. The other points give higher heights, of around 2,900 meters. This means that the hiker should leave the point (0,0) heading towards either of the points $(894,447)$ or $(-894,-447)$. At the end of the half hour, she will be as low as she can be, given her walking speed and the shape of the mountain.

11. For each value of λ the function $h(x,y) = x^2+y^2-\lambda(2x+4y-15)$ has a minimum value $m(\lambda)$.
    (a) Find $m(\lambda)$.
    (b) For which value of λ is $m(\lambda)$ the largest and what is that maximum value?
    (c) Find the minimum value of $f(x,y)=x^2+y^2$ subject to the constraint $2x+4y=15$ using the method of Lagrange multipliers and evaluate λ.
    (d) Compare your answers to parts (b) and (c).

(a) When we are looking for $m(\lambda)$, it means that we can treat λ as a constant in our function, since it will be provided later. That means we need to optimize over x and y, given λ.
There is no (x,y) constraint in this problem, so we simply look for critical points of h, treating λ as a constant.

$$h_x = 2x - 2\lambda$$

$$h_y = 2y - 4\lambda$$

Setting both equal to zero, we get

$$0 = 2x - 2\lambda$$

$$0 = 2y - 4\lambda$$

Solving gives $x = \lambda$

and $y = 2\lambda$

So there is only one critical point, at $(x, y) = (\lambda, 2\lambda)$. We can determine the type of critical point with the second derivative test:

$$\begin{aligned} h_{xx} = 2, h_{yy} = 2, h_{xy} = 0 \\ \text{so } D = (2)(2) - 0^2 = 4 > 0 \\ \text{and } h_{xx} > 0 \text{ (concave up)} \end{aligned}$$

meaning $(x, y) = (\lambda, 2\lambda)$ is a local *minimum*.

To find the actual value of h at the critical point, we sub in the coordinates of the critical point into the original formula:

$$\begin{aligned} h(\lambda, 2\lambda) &= \lambda^2 + (2\lambda)^2 - \lambda(2\lambda + 4(2\lambda) - 15) \\ &= \lambda^2 + 4\lambda^2 - 2\lambda^2 - 8\lambda^2 + 15\lambda \\ &= -5\lambda^2 + 15\lambda \\ &= 5\lambda(3 - \lambda) \\ \text{so } m(\lambda) &= 5\lambda(3 - \lambda) \end{aligned}$$

(b) Now we get to select λ to make m as large as possible. Since m is only a 1D function, we can simply differentiate and set the derivative equal to zero. (Alternatively, we could notice that m is a parabola in λ, and will have its maximum halfway between its roots of $\lambda = 0$ and $\lambda = 3$. That trick only works because m is quadratic, though, so we'll use the more general derivative approach.)

$$\begin{aligned} \frac{dm}{d\lambda} &= -10\lambda + 15 \\ \text{Set derivative equal to zero: } 0 &= -10\lambda + 15 \\ \lambda &= \frac{15}{10} = 1.5 \end{aligned}$$

From m being a quadratic with negative λ^2 coefficient, we know this value of λ gives a maximum of m. The value of $m(1.5) = 11.25$.

(c) Optimize $f(x, y) = x^2 + y^2$ subject to $g(x, y) = 2x + 4y = 15$.

$$\begin{aligned} f_x &= 2x & g_x &= 2 \\ f_y &= 2y & g_y &= 4 \end{aligned}$$

Set up the Lagrange multiplier equations:

$$\begin{aligned} f_x = \lambda g_x \quad &\Rightarrow \quad 2x = 2\lambda && (25) \\ f_y = \lambda g_y \quad &\Rightarrow \quad 2y = 4\lambda && (26) \\ \text{constraint:} \quad &\Rightarrow \quad 2x + 4y = 15 && (27) \end{aligned}$$

Taking (25) / (26), (assuming $\lambda \neq 0$)

$$\begin{aligned} \frac{2x}{2y} &= \frac{\lambda 2}{\lambda 4} \\ \text{so } x &= \frac{y}{2} \end{aligned}$$

Sub into (27) to find

$$2\left(\frac{y}{2}\right) + 4y = 15 \quad \Rightarrow \quad y = 3$$

The value of λ is then $\lambda = x = 1.5$.

Showing that this is a minimum of f requires only noticing that if we move away from this point, $x^2 + y^2$ will grow larger towards infinity. Thus, we must be at a local (and global) minimum of f given the constraint.

(d) We notice that the solutions to both of these problems, (a,b) and (c), are identical. This indicates that there may be alternative ways to set up or interpret constrained optimization problems. The details of these relationships would be included in more advanced math courses like vector calculus and optimization.