Last Name:____________________ **First Name:**____________________

Student Number:____________________

MATH 121 - MIDTERM 1

This test consists of 4 questions to be answered in the space provided. Show all work and give explanations when needed.

1. (a) What is the derivative of $f(x) = x^3 + 2x + 7$? (2 marks)
   (b) Verify your answer from part (a) by calculating the derivative using the **limit definition**. (3 marks)
   (c) What is the **2nd order** Taylor polynomial for $f(x)$ at $x = 1$? (5 marks)

a) use power rule: $f'(x) = 3x^2 + 2$

b) limit definition: $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$$f'(x) = \lim_{\Delta x \to 0} \Big[(x+\Delta x)^3 + 2(x+\Delta x) + 7 \Big] - \Big[x^3 + 2x + 7 \Big] \Big/ \Delta x$$

$$= \lim_{\Delta x \to 0} \left(\Big[\cancel{x^3} + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + \cancel{2x} + 2\Delta x + \cancel{7} \Big] - \cancel{x^3} - \cancel{2x} - \cancel{7} \right) \Big/ \Delta x$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 \cancel{\Delta x} + 3x(\Delta x)^{\cancel{2}\,1} + (\Delta x)^{\cancel{3}\,2} + 2\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2 + 2 \right)$$

now plug in $\Delta x = 0$

$$= 3x^2 + 2$$

c) 2nd order Taylor polynomial:

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$a = 1.$

$f(1) = (1)^3 + 2(1) + 7 = 10$

$f'(x) = 3x^2 + 2$

$f'(1) = 3(1)^2 + 2 = 5$

$f''(x) = 6x$

$f''(1) = 6$

$\therefore \quad P_2(x) = 10 + 5(x-1) + 3(x-1)^2$

2. Consider the function $f(x) = \left(x^2 - 4x + 5\right) e^{-x}$, for $x \geq 0$.

   (a) What is $f(0)$? (1 mark)

   (b) What is $\lim_{x\to\infty} f(x)$? (No calculations necessary) (1 mark)

   (c) Calculate the first and second derivatives: $f'(x)$ and $f''(x)$. (4 marks)

   (d) Using your previous answers, sketch $f(x)$ for $x \geq 0$. Label the y-intercept, and any critical or inflection points. (hint: remember to factor your answers for $f'(x)$ and $f''(x)$, this will help you to draw sign diagrams). (4 marks)

a) $f(0) = (0 - 0 + 5)\, e^{-(0)} = (5)(1) = 5$

b) $\lim_{x\to\infty} f(x) = 0$

(because e^{-x} is dominant term, but you dont have to justify)

c) use product rule:

$$f'(x) = (x^2 - 4x + 5)(-e^{-x}) + (2x - 4)(e^{-x})$$

$$= (-x^2 + 4x - 5 + 2x - 4)\, e^{-x}$$

$$= (-x^2 + 6x - 9)\, e^{-x}$$

$$= -(x-3)^2 e^{-x}$$

$$F''(x) = \frac{d}{dx}\left((-x^2 + 6x - 9) e^{-x} \right)$$

$$= (-x^2 + 6x - 9)(-e^{-x}) + (-2x + 6)(e^{-x})$$

c) contd

$$f''(x) = (x^2 - 6x + 9 - 2x + 6)e^{-x}$$

$$= (x^2 - 8x + 15)e^{-x}$$

$$= (x-3)(x-5)e^{-x}$$

d) sign diagrams : (look at the signs of factors)

$f'(x)$

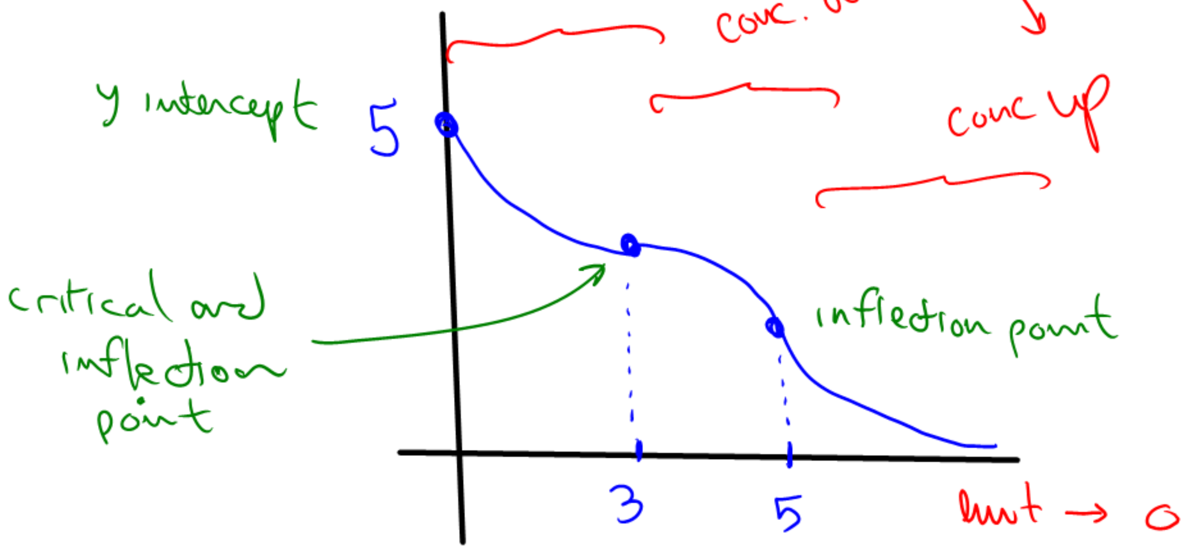
$-$	0	$-$
	$x=3$	

always decreasing

$f''(x)$

$+$	0	$-$	0	$+$
	$x=3$		$x=5$	

conc up. conc. down conc up



3. If an electrical circuit consists of two resistors R_1 and R_2 connected in parallel, then the total resistance R of the circuit is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

All resistances are measured in ohms (Ω). Assume R_2 has a **constant** resistance equal to 5 Ω, but R_1 is a **variable** resistor.

(a) Find $\frac{dR}{dR_1}$. What are the units of this quantity? (4 marks)

(b) Suppose the current resistance of R_1 is 1 Ω and it is is increasing at a rate of 0.2 Ω/sec. How fast is the total resistance of the circuit increasing? State your answer with appropriate units. (6 marks)

a) two ways to answer this one :

method 1

implicit differentiation $\frac{d}{dR_1}\left(\frac{1}{R}\right) = \frac{d}{dR_1}\left(\frac{1}{R_1} + \frac{1}{5}\right)$

$$\frac{-1}{R^2} \cdot \frac{dR}{dR_1} = -\frac{1}{R_1^2}$$

$$\frac{dR}{dR_1} = \frac{R^2}{R_1^2}$$

units $= \frac{\text{units of } R}{\text{units of } R_1} = \frac{\cancel{\Omega}}{\cancel{\Omega}} =$ dimensionless ratio

a) cont'd

method 2

Solve for R first.

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

$$\therefore \quad R = \frac{R_1 R_2}{R_1 + R_2}$$

now differentiate normally using quotient rule

$$\frac{dR}{dR_1} = \frac{(R_1+R_2)(R_2) - (R_1 R_2)(1)}{(R_1+R_2)^2}$$

$$= \frac{\cancel{R_1 R_2} + R_2^2 - \cancel{R_1 R_2}}{(R_1+R_2)^2}$$

$$= R_2^2 / (R_1+R_2)^2$$

(looks different than the other answer... but

$$\frac{R^2}{R_1^2} = \frac{\left(\frac{R_1 R_2}{R_1+R_2}\right)^2}{R_1^2} = R_2^2/(R_1+R_2)^2$$, actually the same)

b) related rates problem, use the chain rule ...

$$\frac{dR}{dt} = \frac{dR}{dR_1} \cdot \frac{dR_1}{dt}$$

$$= \frac{R^2}{R_1^2} \cdot \frac{dR_1}{dt}$$

$\left(\text{or } \frac{R_2^2}{(R_1+R_2)^2} \right)$

$R_1 = 1\ \Omega \qquad R_2 = 5\ \Omega$

$$\frac{dR_1}{dt} = .2\ \Omega/\text{sec} = \frac{1}{5}\ \Omega/\text{sec}$$

$$R = \frac{R_1 R_2}{R_1 + R_2} = \frac{(1)(5)}{(1)+(5)} = 5/6\ \Omega$$

$$\Rightarrow \quad \frac{dR}{dt} = \frac{(5/6)^2}{(1)^2} \cdot (1/5)$$

$$= 5/36\ \Omega/\text{sec}$$

4. You are constructing a rectangular storage shed with a **square** base. You need space to accomodate 75 m^3 of stuff. The floor is made of concrete which costs 2 $\$/m^2$. The four walls are made of wood, which costs 5 $\$/m^2$. The roof tiles cost 4 $\$/m^2$.

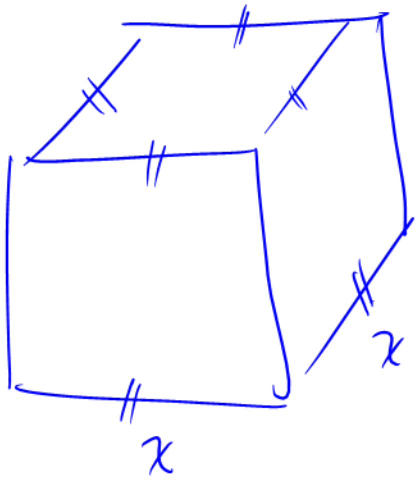
   (a) Write a formula for the total cost of the shed in terms of a single variable. Indicate what this variable represents, and state its domain. (3 marks)

   (b) What should be the dimensions of the shed to **minimize** the cost? State the length, width and height, and also the minimum value of the cost. (5 marks)

   (c) Suppose you only have enough concrete to cover an area of 16 m^2. How does this **change** the domain of the variable you chose in part (a)? (1 mark)

   (d) With the restriction from part (c), can you achieve the minimum cost you found before? If not, what is the new minimum cost? (1 mark)

a)



x = length and width of base

domain: $x \in (0,\infty)$

total cost = (floor) + (roof) + (4 sides)

$$C(x,y) = 2x^2 + 4x^2 + 4(5xy)$$

$$= 6x^2 + 20xy$$

Volume $V = x^2 y = 75$

$\therefore \quad y = 75/x^2$

substitute in formula for $C(x,y)$ ...

a) cont'd

$$C(x) = 6x^2 + 20x\left(\frac{75}{x^2}\right)$$

$$C(x) = 6x^2 + \frac{1500}{x}$$

b) optimize ... solve $C'(x) = 0$

$$C'(x) = 12x - \frac{1500}{x^2} = 0$$

$$12x = \frac{1500}{x^2}$$

$$12x^3 = 1500$$

$$x^3 = 125$$

$$x = 5$$

$$y = \frac{75}{x^2} = \frac{75}{(5)^2} = 3$$

dimensions are 5 m x 5 m x 3 m

b) cont'd

minimum cost is $C(5) = 6(5)^2 + \frac{1500}{(5)}$

$= 150 + 300$

$= \$450$

(you dont have to verify its a minimum using 1st / 2nd derivative tests)

c) the base cannot be bigger than $16\ m^2 = 4\ m \times 4\ m$

$\therefore$ domain of x is now $(0,4]$

d) our minimum cost was with $x = 5\ m$, base is $25\ m^2$ ... not in domain!

the global minimum must be at the end point $x = 4\ m$ instead.

new minimum cost $C(4) = \$471$

(extra space)