Unit #9 : Definite Integral Properties; Fundamental Theorem of Calculus

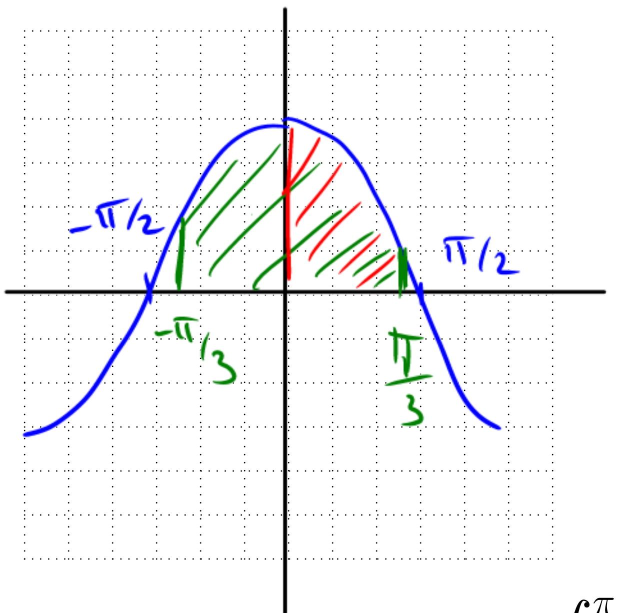
Goals:

- Identify properties of definite integrals
- Define odd and even functions, and relationship to integral values
- Introduce the Fundamental Theorem of Calculus
- Compute simple anti-derivatives and definite integrals

Reading: Textbook reading for Unit #9 : Study Sections 5.4, 5.3, 6.2

Properties of Definite Integrals

Example: *Sketch the area implicit in the integral* $\displaystyle\int_{-\pi/3}^{\pi/3} \cos(x)\ dx$



If you were told that $\displaystyle\int_{0}^{\pi/3} \cos(x)\ dx = \frac{\sqrt{3}}{2}$, *find the size of the area you sketched.*

$= 2 \times$ $= 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$ sq. units

by symmetry

This example highlights an important and intuitive general property of definite integrals.

Additive Interval Property of Definite Integrals

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

$$\int_{-\pi/3}^{0} + \int_{0}^{\pi/3} = \int_{-\pi/3}^{\pi/3}$$

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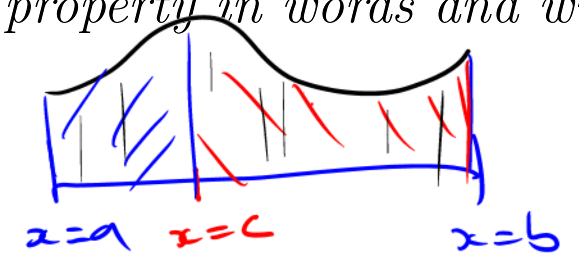
Explain this general property in words and with a diagram.



$$\int_{x=a}^{x=b} = \int_{x=a}^{x=c} + \int_{x=c}^{x=b}$$

A more rarely helpful, but equally true, corollary of this property is a second property:

Reversed Interval Property of Definite Integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Use the integral $\int_0^{\pi/3} \cos(x)\,dx + \int_{\pi/3}^{0} \cos(x)dx$*, and the earlier interval property, to illustrate the reversed interval property.*

$$\int_{a=0}^{c=\pi/3} \cos(x)\,dx + \int_{c=\pi/3}^{b=0} \cos(x)\,dx = \int_{a=0}^{b=0} \cos(x)\,dx = 0$$

interval from $x=0$ to $x=0$

$\Rightarrow$ zero width

$\Rightarrow$ zero area

$$\int_0^{\pi/3} \cos(x)\,dx = -\int_{\pi/3}^{0} \cos(x)\,dx$$

Give a rationale related to Riemann sums for the Reversed Interval property.

$$\int_a^b f(x)\,dx \overset{\text{def'n}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \Delta x = \frac{b-a}{n}$$

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_2 \qquad \Delta x_2 = \frac{a-b}{n} = -1\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(-\Delta x)$$

$$= -\underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x}$$

$$= -\int_a^b f(x)\,dx$$

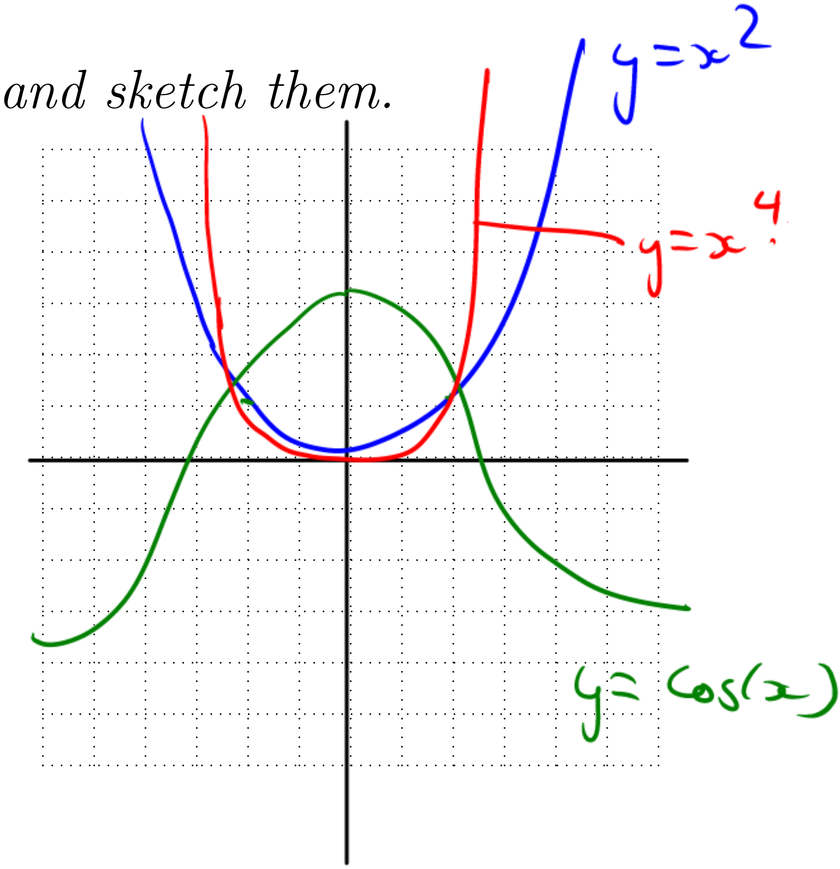
Even and Odd Functions

These properties can be helpful especially when dealing with *even* and *odd* functions.

Define an **even** *function. Give some examples and sketch them.*

$f(x) = f(-x)$

symetric across $x=0$



$y = x^2 + 5\cos(x)$

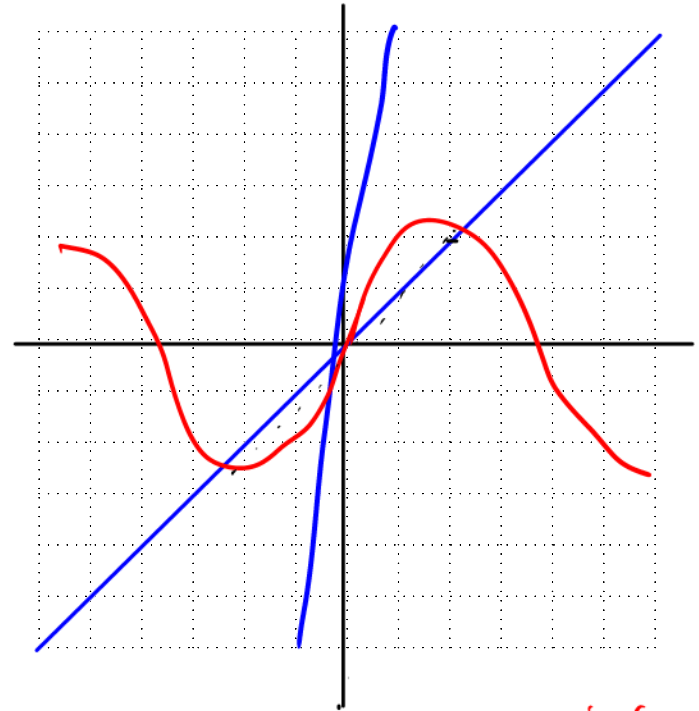
is also even.

Define an **odd** *function. Give some examples and sketch them.*

$f(x) = -f(-x)$

symmetric through $(0,0)$

sums of odd functions are also odd



$y = x$ $\quad$ $y = \sin(x)$

$y = 10x$ $\quad$ $y = x^3$

Integral Properties of Even and Odd Functions

Find a property of **odd** *functions when you integrate on both sides of* $x = 0$.

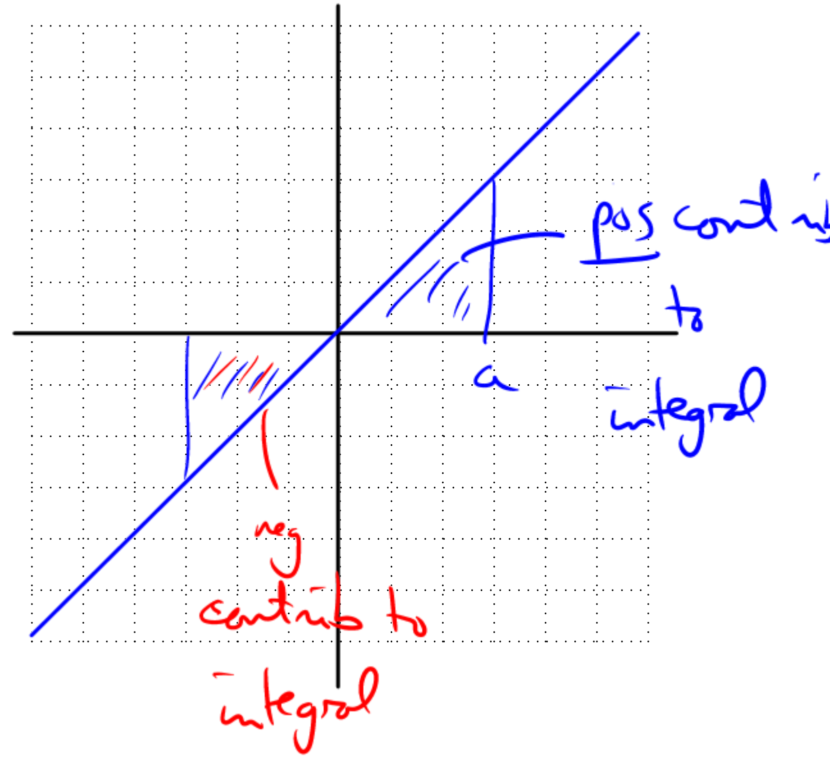
areas are of identical size

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$$

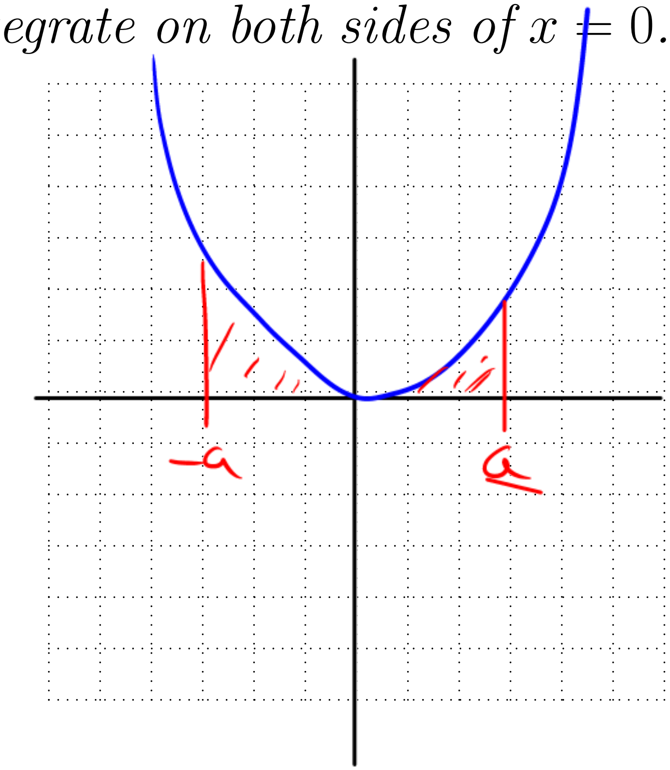
for any odd function $f(x)$

$$\int_{-a}^{0} f(x)\,dx + \underbrace{\int_{0}^{a} f(x)\,dx}$$

equal in size, but opposite in sign



Find a property of **even** *functions when you integrate on both sides of* $x = 0$.

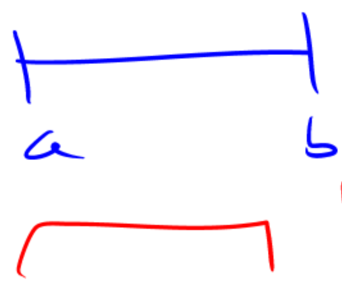


$$\int_{-a}^{a} g(x)\,dx = 2 \times \int_{0}^{a} g(x)\,dx$$

b/c of symmetry if $g(x)$ is an <u>even</u> function.

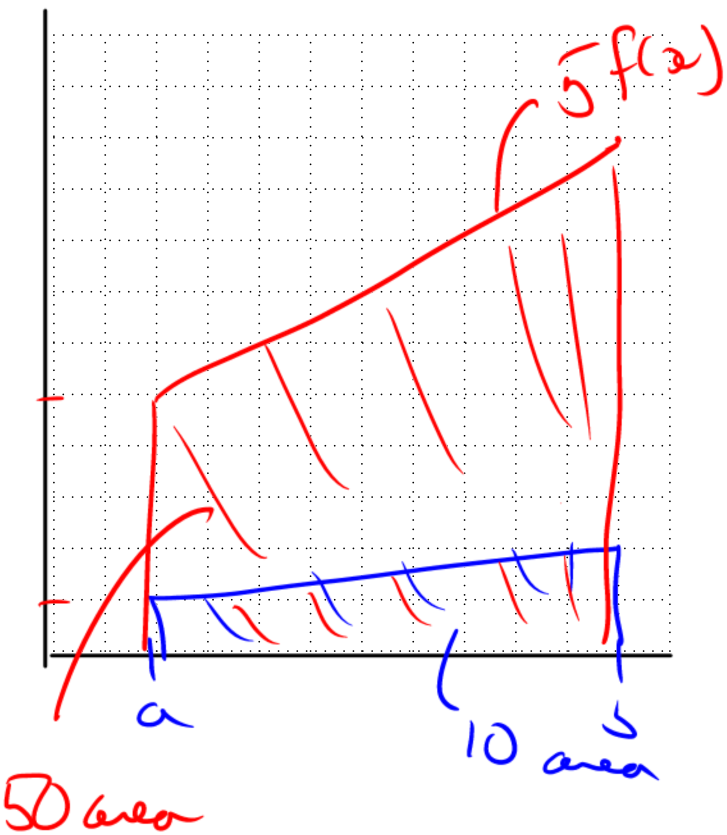
Linearity of Definite Integrals

Example: *If* $\displaystyle\int_a^b f(x)\ dx = 10$, *then what is the value of* $\displaystyle\int_a^b 5f(x)\ dx$*?* $= 5 \cdot 10 = 50$
Sketch an area rationale for this relation.



bring consts out front of integrals

$$\int_a^b 5f(x)dx = 5\int_a^b f(x)dx$$

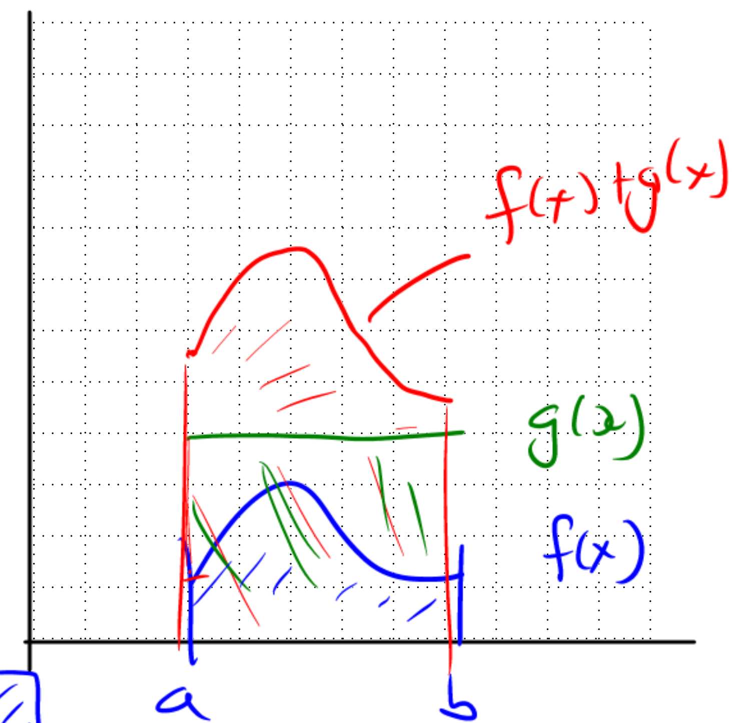


vertical scaling
⇒ stretch f(x) vertically by 5
⇒ area represented by the integral will inc. by a factor of 5

Example: *If* $\displaystyle\int_a^b f(x)\,dx = 2$, *and* $\displaystyle\int_a^b g(x)\,dx = 4$ *then what is the value of* $\displaystyle\int_a^b \left(f(x) + g(x)\right) dx$*? Again, sketch an area rationale for this relation.*

$= 2 + 4 = 6$

$$\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



Linearity of Definite Integrals

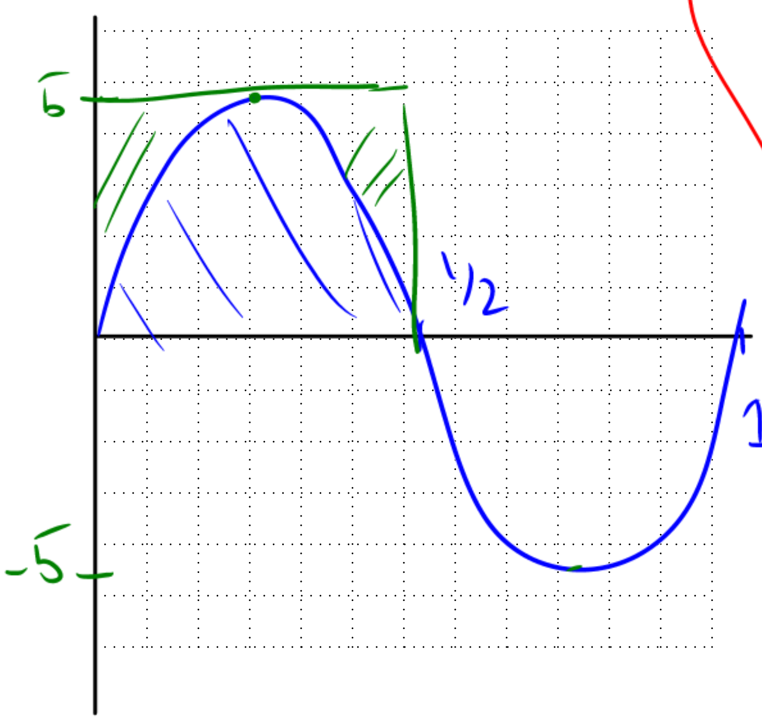
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Simple Bounds on Definite Integrals

Example: *Sketch a graph of* $f(x) = 5\sin(2\pi x)$, *then use it to make an area argument proving the statement that*

$$0 \leq \int_0^{\frac{1}{2}} 5\sin(2\pi x)\, dx \leq \frac{5}{2}$$



amplitude

period is $\frac{2\pi}{2\pi} = 1$

$$\int_0^{1/2} 5\sin(2\pi x)\, dx \leq \boxed{}$$

area $5/2$

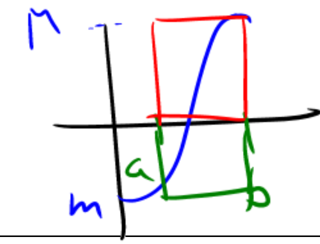
$5\sin(2\pi x) \geq 0$ on $0 \leq x \leq 1/2$

so $0 \leq \int_0^{1/2} 5\sin(2\pi x)\, dx$

area

Simple Maximum and Minimum Values for Definite Integrals
If a function $f(x)$ is continuous and bounded between $y = m$ and $y = M$ on the interval $[a, b]$, i.e. $m \leq f(x) \leq M$ on the interval, then

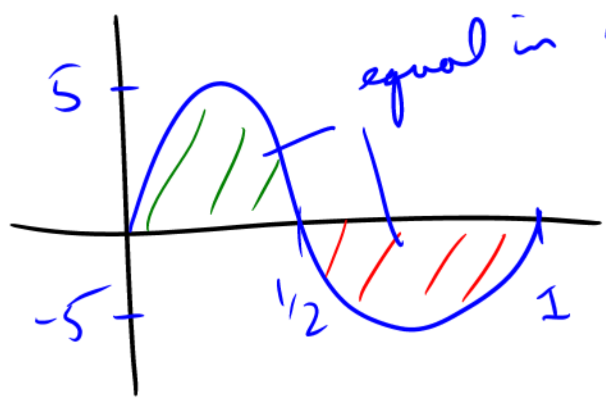
$$m(b - a) \leq \int_a^b f(x)\, dx \leq M(b - a)$$



Note that the maximum and minimum values we get with the method above are quite crude. Sometimes you will be asked for much more precise values which can often be just as easy to find.

Example: *Use the graph to find the* exact *value of* $\int_0^1 5\sin(2\pi x)\, dx$ *i.e. not just a range, but the single correct area value.*

area equal in magnitude



so $\displaystyle\int_0^1 5\sin(2\pi x)\,dx$

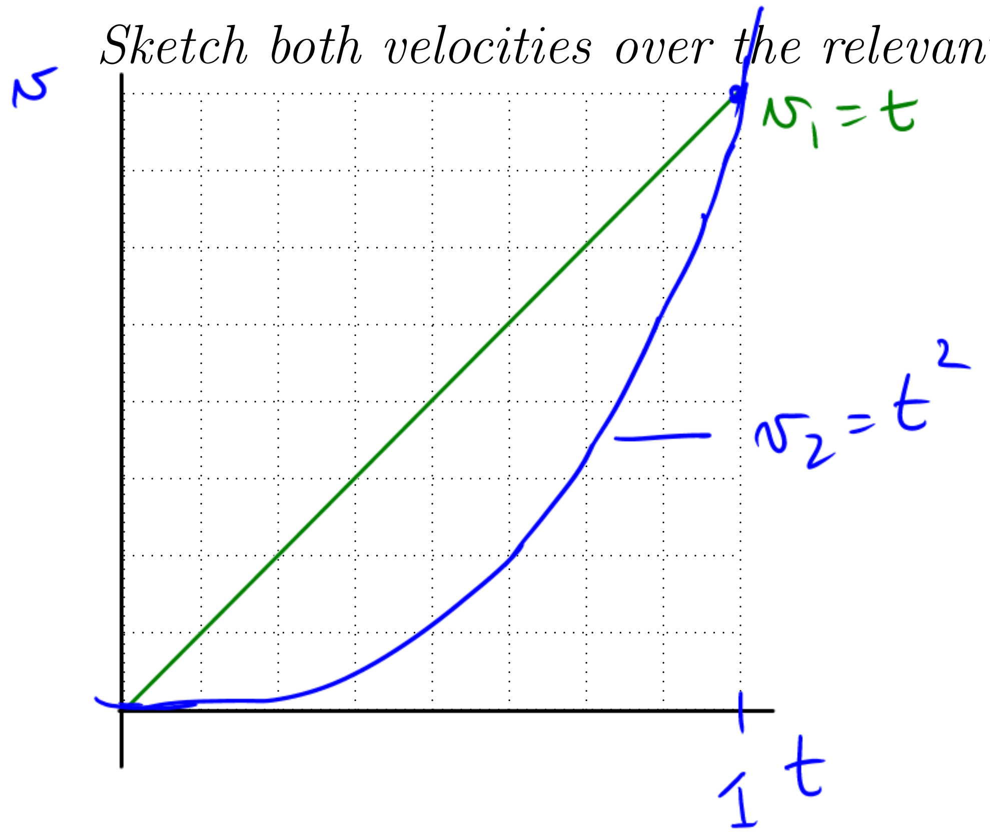
$= 0$ by symmetry in sine func.

Relative Sizes of Definite Integrals

Example: *Two cars start at the same time from the same starting point. For the first second,*

- *the first car moves at velocity* $v_1 = t$, *and*
- *the second car moves at velocity* $v_2 = t^2$.

Sketch both velocities over the relevant interval.



Which car travels further in the first second? Relate this to a definite integral.

$v_1 \geq v_2$ on $t \in [0,1]$ $\Rightarrow$ car 1 is moving faster (or same speed)

$\Rightarrow$ car 1 travelled furthest in $t = [0,1]$.

$$v_1 \geq v_2 \Rightarrow \int_0^1 v_1(t)\,dt \geq \int_0^1 v_2(t)\,dt$$

Comparison of Definite Integrals

If $f(x) \leq g(x)$ on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

The Fundamental Theorem of Calculus

Reading: Section 5.3 and 6.2

We have now drawn a firm relationship between area calculations (and physical properties that can be tied to an area calculation on a graph). The time has now come to build a method to compute these areas in a systematic way.

The Fundamental Theorem of Calculus

If f is continuous on the interval $[a, b]$, and we define a related function $F(x)$ such that $F'(x) = f(x)$, then

new F we need to find

original

plug in b, a into this magic function F(x)

$$\underbrace{\int_a^b f(x)\,dx}_{\text{complicated area-related calculation}} = \overbrace{F(b) - F(a)}$$

The fundamental theorem ties the *area* calculation of a definite integral back to our earlier *slope* calculations from derivatives. However, it changes the direction in which we take the derivative:

$f(x) \to f'(x)$

Units 1→7

- Given $f(x)$, we find the *slope* by finding the *derivative* of $f(x)$, or $f'(x)$.

Units 8, up

- Given $f(x)$, we find the *area* $\displaystyle\int_a^b f(x)\, dx$ by finding $F(x)$ which is the *anti-derivative* of $f(x)$; i.e. a function $F(x)$ for which $F'(x) = f(x)$.

$F' \to F$

In other words, if we can find an anti-derivative $F(x)$, then calculating the value of the definite integral requires a simple evaluation of $F(x)$ at two points $(F(b) - F(a))$. This last step is *much* easier than computing an area using finite Riemann sums, and also provides an exact value of the integral instead of an estimate.

Example: *Use the Fundamental Theorem of Calculus to find the area bounded by the x-axis, the line $x = 2$, and the graph $y = x^2$. Use the fact that* $\dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) = x^2$.

$\frac{1}{3}3x^2$

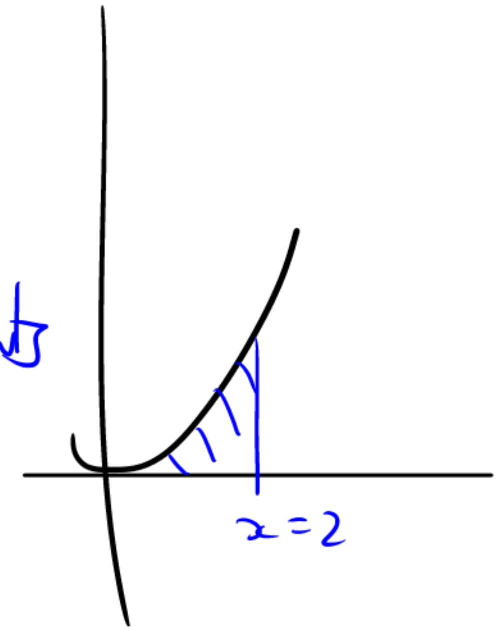
by the F.T.C.

$F(2) - F(0)$

$$\int_0^2 \underbrace{x^2}_{f(x)}\,dx = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3} \text{ sq units}$$

and $F(x) = \frac{1}{3}x^3$

b/c $F'(x) = x^2 = f(x)$



Sketch the area interpretation of this result.

anti-deriv

We used the fact that $F(x) = \frac{1}{3}x^3$ *is an anti-derivative of* x^2*, so we were able use the Fundamental Theorem.*

deriv

Give another function $F(x)$ *which would also satisfy* $\frac{d}{dx}F(x) = x^2$.

anti deriv

$F(x) = \frac{1}{3}x^3 + \{\text{any const}\}$ $\quad f(x) = x^2 + 0$

d/dx

Use the Fundamental Theorem again with this new function to find the area implied by $\int_0^2 x^2\,dx$.

check

eg. $F(x) = \frac{1}{3}x^3 + 17 \rightarrow F'(x) = \frac{1}{3}(3x^2) + 0 = x^2$ ✓

$= f(x)$

$F(2) - F(0)$

$$\int_0^2 x^2\,dx = \left(\frac{1}{3}2^3 + 17\right) - \left(\frac{1}{3}0^3 + 17\right)$$

$= \frac{8}{3}$ sq units, as before

Did the area/definite integral value change? Why or why not?

No: the $F(b) - F(a)$ form in the F.T.C. guaranteed that any (+ const) will cancel out in the final evaluation.

Based on that result, give the most general version of $F(x)$ you can think of. Confirm that $\dfrac{d}{dx}F(x) = x^2$.

$$F(x) = \frac{1}{3}x^3 + C \quad \downarrow d/dx$$

$$\text{b/c} \quad F'(x) = \frac{1}{3}(3x^2) + 0$$

$$= x^2$$ regardless of value of C.

With our extensive practice with derivatives earlier, we should find it straightforward to determine some simple *anti*-derivatives.
Complete the following table of anti-derivatives.

function $f(x)$	anti-derivative F(x)
x^2	$\frac{x^3}{3} + C$
x^n	$\frac{x^{n+1}}{n+1} + C$
$x^2 + 3x^1 - 2x^0$	$\frac{x^3}{3} + 3\frac{x^2}{2} - 2\frac{x^1}{1} + C$

d/dx

d/dx

$x^2, x, x^{3/2}, x^{-1/2}$

x in base

use linearity of integrals

always verify an anti-deriv by differentiating

add one to the power divide by the new power

function $f(x)$	anti-derivative F(x)
$\cos x$	$\sin(x) + C$
	d/dx
$\sin x$	$-\cos(x) + C$
	d/dx
$x + \sin x$	$\frac{x^2}{2} + (-\cos(x)) + C$

linearity

function $f(x)$	anti-derivative F(x)
e^x	$e^x + C$
$\frac{\cancel{\ln(2)}}{\cancel{\ln(2)}} \cdot 2^x$	$\frac{2^x}{\ln(2)} + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$
$\frac{1}{x}$	$\ln(x) + C$

d/dx

d/dx

d/dx

x in exponent

will revisit

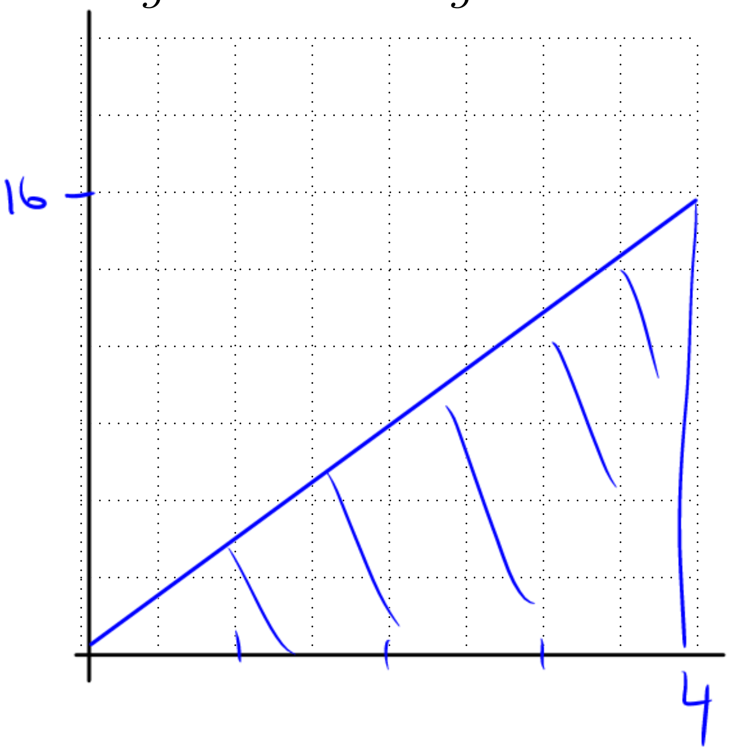
from page 31

$\ln(|x|) + C$

The chief importance of the **Fundamental Theorem of Calculus (F.T.C.)** is that it enables us (potentially at least) to find values of definite integrals more accurately and more simply than by the method of calculating Riemann sums. In principle, the F.T.C. gives a precise answer to the integral, while calculating a (finite) Riemann sum gives you no better than an approximation.

LEFT, MID

Example: *Consider the area of the triangle bounded by* $y = 4x$*,* $x = 0$ *and* $x = 4$*. Compute the area based on a sketch, and then by constructing an integral and using anti-derivatives to compute its value.*



area $\boxed{}$ by triangle $= \frac{1}{2} b \cdot h = \frac{1}{2}(4)(16)$

$= 32$ sq units

$f(x) \longrightarrow F(x) = 4\frac{x^2}{2} + C$

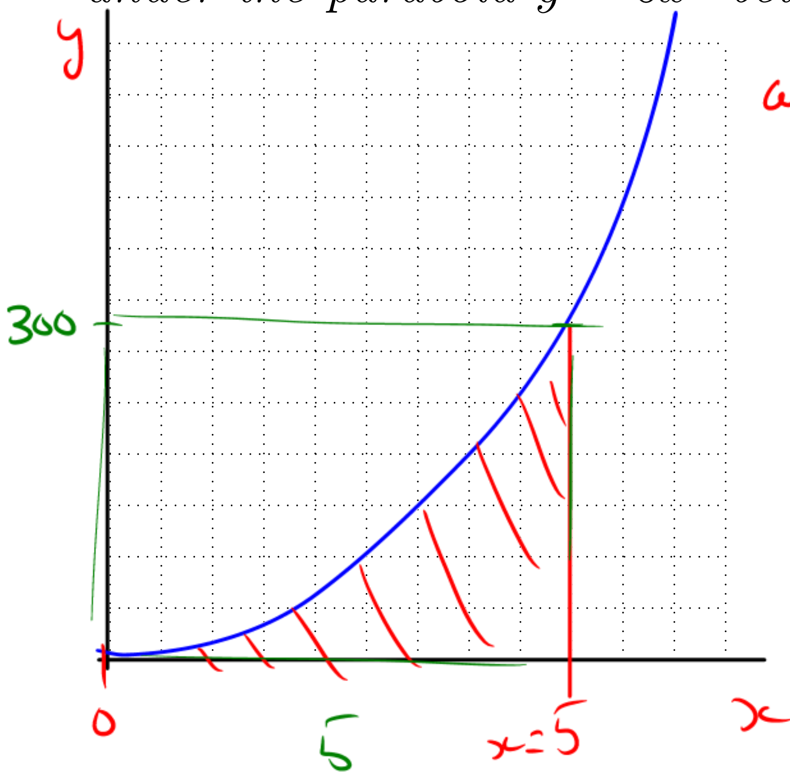
$$\underbrace{\int_0^4 4x^1\,dx}_{\text{area under } y=4x \text{ b/w } x=0 \text{ and } x=4} \underset{\text{the F.T.C}}{=} F(4) - F(0)$$

$$= \left(\frac{4(4^2)}{2} + \cancel{C}\right) - \left(\frac{4(0)^2}{2} + \cancel{C}\right)$$

$$= 2 \cdot 16 = 32 \text{ sq units}$$

which matches our direct area calc'n ✓

Example: *Use a definite integral and anti-derivatives to compute the area under the parabola $y = 6x^2$ between $x = 0$ and $x = 5$.*



area cannot be found exactly w/ simple shapes

but area can be found w/ F.T.C.

$$f(x) \longrightarrow F(x) = 6\frac{x^3}{3} + C$$

$$\int_0^5 6x^2\,dx = F(5) - F(0)$$

by F.T.C.

$$= \left(\frac{6(5)^3}{3} + \cancel{C}\right) - \left(\frac{6\cdot 0^3}{3} \cancel{+C}\right)$$

$$= 250 \text{ sq units}$$

area $= 300 \cdot 5$

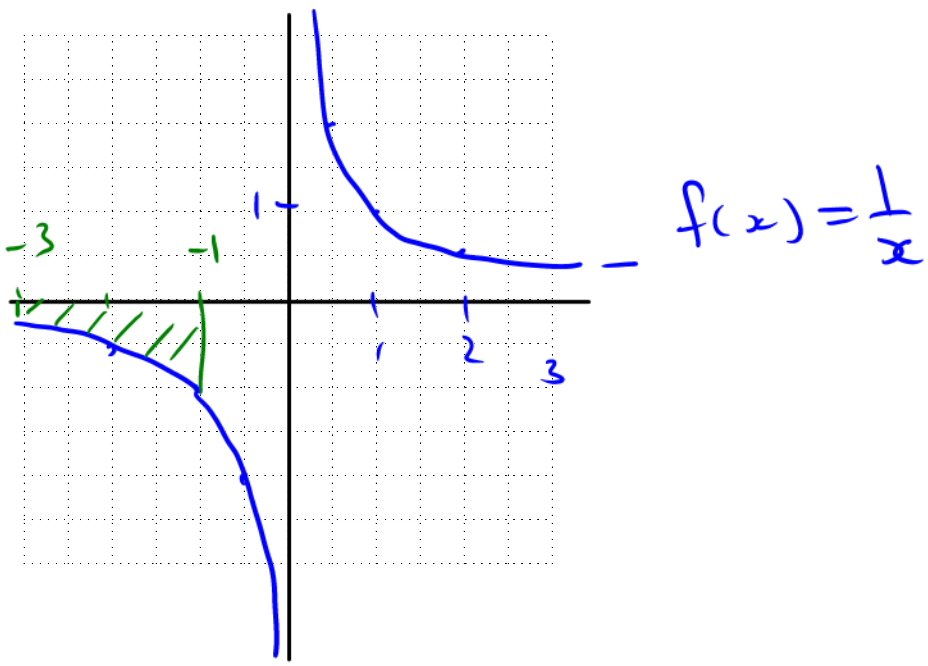
$= 1500$ sq units

$f(x) = \frac{1}{x} \to F(x) = \ln(x) + C$

$\frac{d}{dx}$

The last entry in our anti-derivative table was $f(x) = \dfrac{1}{x}$. It is a bit of a special case, as we can see in the following example.

Example: *Sketch the area implied by the integral* $\displaystyle\int_{-3}^{-1} \frac{1}{x}\, dx$.

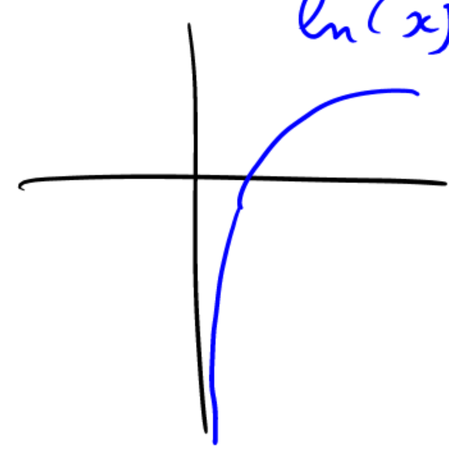


Example: *Now use the anti-derivative and the Fundamental Theorem of Calculus to obtain the exact area under* $f(x) = \dfrac{1}{x}$ *between* $x = -3$ *and* $x = -1$*. Make any necessary adaptations to our earlier anti-derivative table.*

$$\int_{-3}^{-1} \underbrace{\frac{1}{x}}_{f(x)} dx = F(-1) - F(-3)$$

↳ $F(x) = \ln(x) + C$

↳ $= (\underbrace{\ln(-1)}_{\text{not defined}} + \cancel{C}) - (\underbrace{\ln(-3)}_{\text{not defined}} + \cancel{C})$??



Challenge: need a better / more general antiderivative

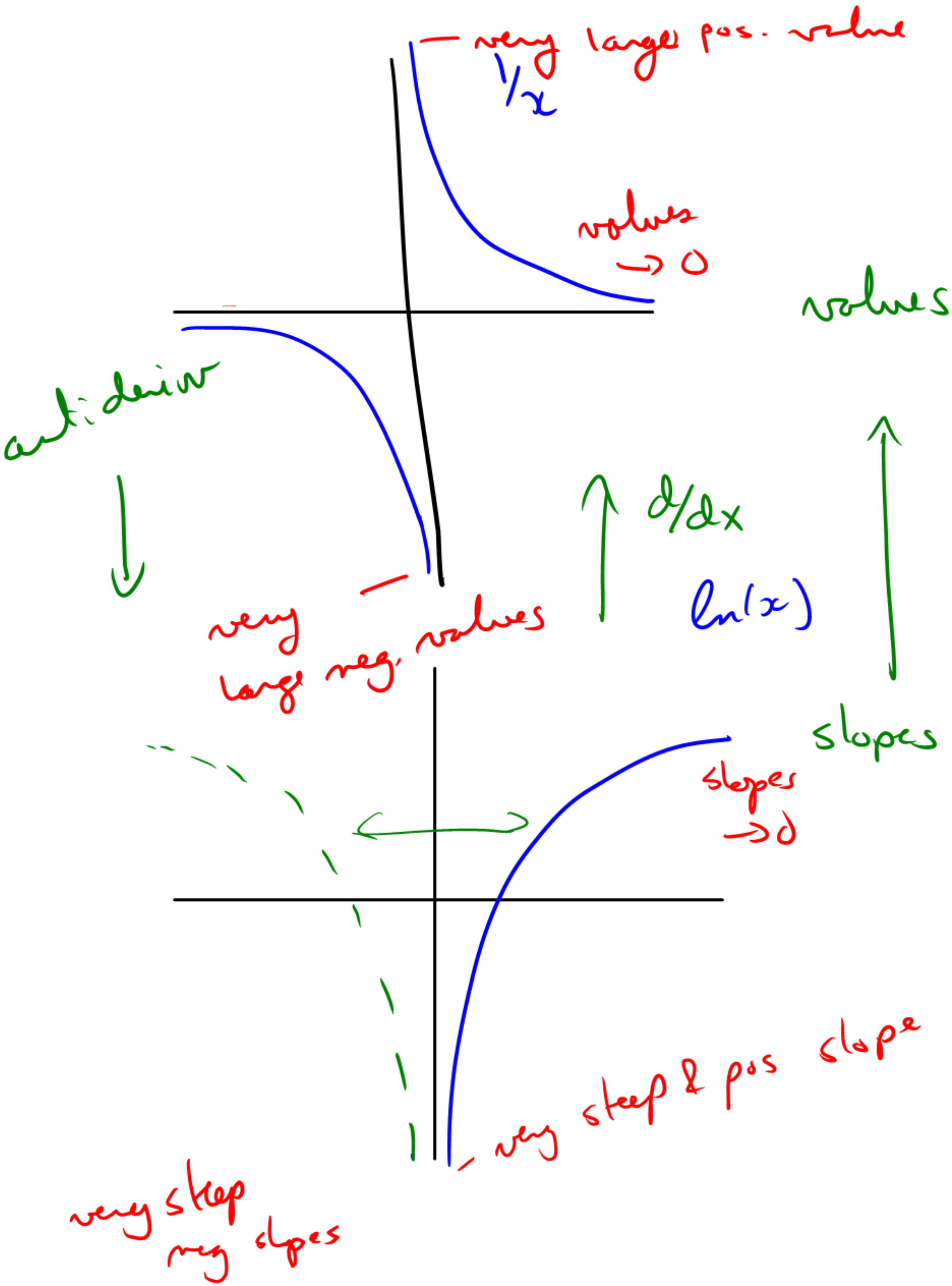
for $\frac{1}{x}$; · $\frac{1}{x}$ defined for all $x \neq 0$

· but $\ln(x)$ only defined for $x > 0$

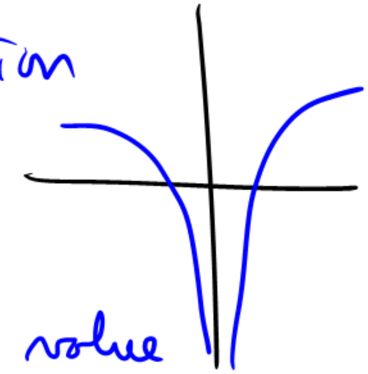
with new antideriv from next page

$= (\ln|-1| + \cancel{C}) - (\ln(|-3|) + \cancel{C}) = \ln(1) - \ln(3)$

$= -1.0986$ (neg ✓)



New function



- same y value for both x and $(-x)$ (even)

$$\boxed{y = \ln(|x|)}$$

and

the anti-deriv of $\frac{1}{x}$ for all $x \neq 0$ will be $\ln(|x|) + C$

Anti-derivatives and the Fundamental Theorem of Calculus

The F.T.C. tells us that if we want to evaluate

$$\int_a^b f(x)\,dx$$

all we need to do is find an anti-derivative $F(x)$ of $f(x)$ and then evaluate $F(b) - F(a)$.

THERE IS A CATCH. While in many ~~some~~ cases this really is very clever and straightforward, in other cases finding the anti-derivative can be surprisingly difficult. This week, we will stick with simple anti-derivatives; in later weeks we will develop techniques to find more complicated anti-derivatives.

Some general remarks at this point will be helpful.

Remark 1

definite integral

Because of the importance of finding an anti-derivative of $f(x)$ when you want to calculate $\int_a^b f(x)\,dx$, it has become customary to denote the anti-derivative itself by the symbol

$$\int f(x)\,dx \quad = \text{anti deriv of } f(x)$$

like $\frac{d}{dx} f(x) = \text{deriv of } f(x)$

The symbol $\int f(x)\,dx$ (with no limits on the integral) refers to the anti-derivative(s) of $f(x)$, and is called the **indefinite integral** of $f(x)$

→ "area"

Note that the definite integral is a number, but the indefinite integral is a function (really a family of functions).

Remark 2

Since there are always infinitely many anti-derivatives, all differing from each other by a constant, we customarily write the anti-derivative as a family of functions, in the form $F(x) + C$. For example,

$$\int x^2\, dx = \frac{x^3}{3} + \underline{\underline{C}}$$

Note that *an anti-derivative* is a single function, while the *indefinite integral* is a family of functions.

- always include +C
  w/ indefinite integral

Remark 3

Since the last step in the evaluation of the integral $\int_a^b f(x)\,dx$, once the anti-derivative $F(x)$ is found, is the evaluation $F(b) - F(a)$, it is customary to write $F(x)\Big|_a^b$ in place of $F(b) - F(a)$, as in

$$\int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = \frac{4^3}{3} - \frac{0^3}{3}$$

+C not necessary for definite integrals b/c +C − (+C) always cancel

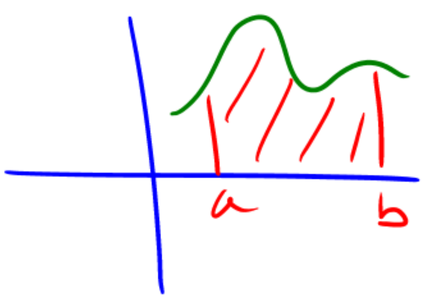
F(b) − F(a)

anti-deriv / integration step

Remark 4

The variable x in the definite integral $\int_a^b f(x)\,dx$ is called the *variable of integration*. It can be replaced by another variable name without altering the value of the integral.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(\theta)\,d\theta$$



x (or u, or θ)