Unit #10 : Graphs of Antiderivatives; Substitution Integrals

Goals:

- Relationship between the graph of f(x) and its anti-derivative F(x)
- The guess-and-check method for anti-differentiation.
- The substitution method for anti-differentiation.

Reading: Textbook reading for Unit #10 : Study Sections 6.1, 7.1

The Relation between the Integral and the Derivative Graphs

the F.T.C.

We saw last week that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

if $F(x)$ is an *anti*-derivative of $f(x)$. or some $F(x)$ that satisfies $F'(x) = f(x)$

Recognizing that finding anti-derivatives would be a central part of evaluating integrals, we introduced the notation

$$\int f(x)\,dx = F(x) + C \Leftrightarrow F'(x) = f(x)$$

Many times when we can't easily evaluate or find an anti-derivative by hand, we can at least sketch what the anti-derivative would look like; there are very clear relationships between the graph of $f(x)$ and its anti-derivative $F(x)$.

Example: *Consider the graph of $f(x)$ shown below. Sketch two possible anti-derivatives of $f(x)$.*

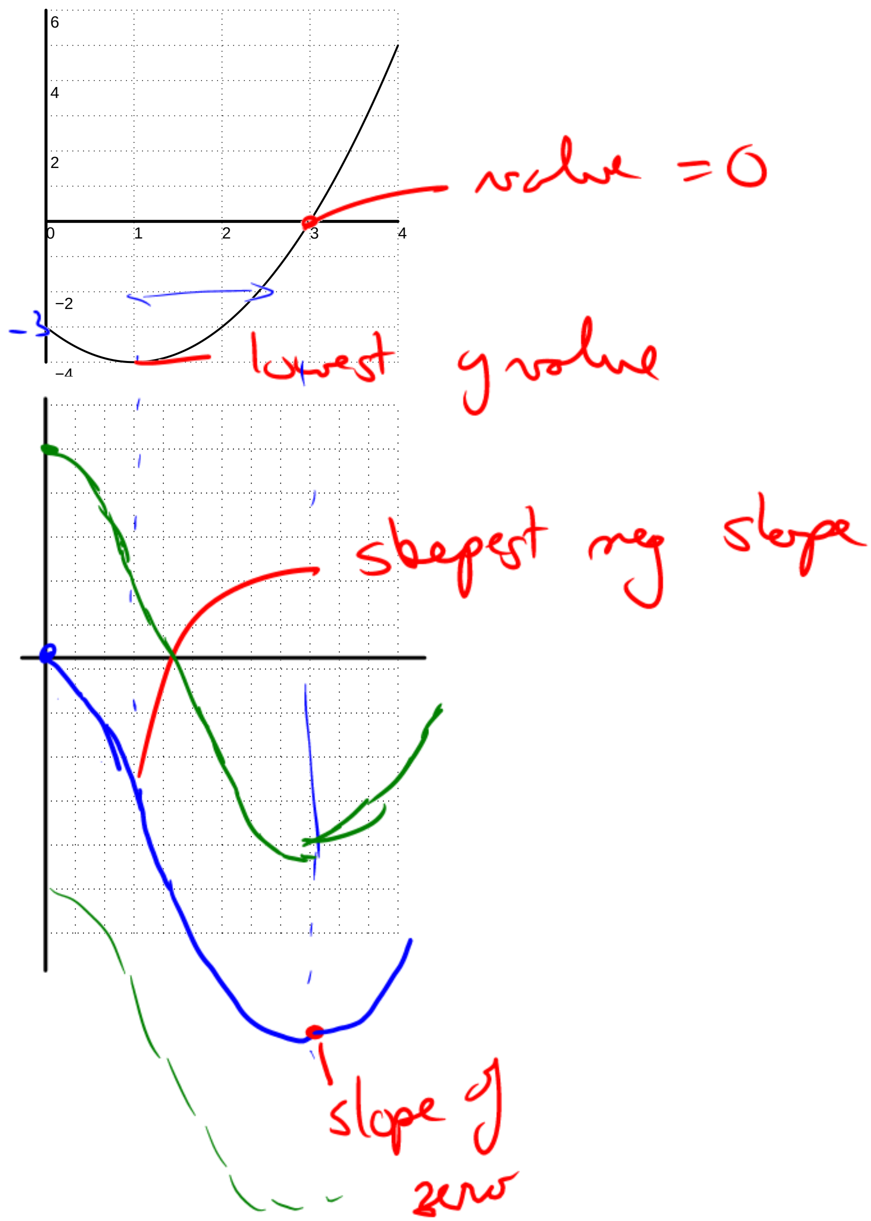
(y)-values

slopes

f(x)

F(x)

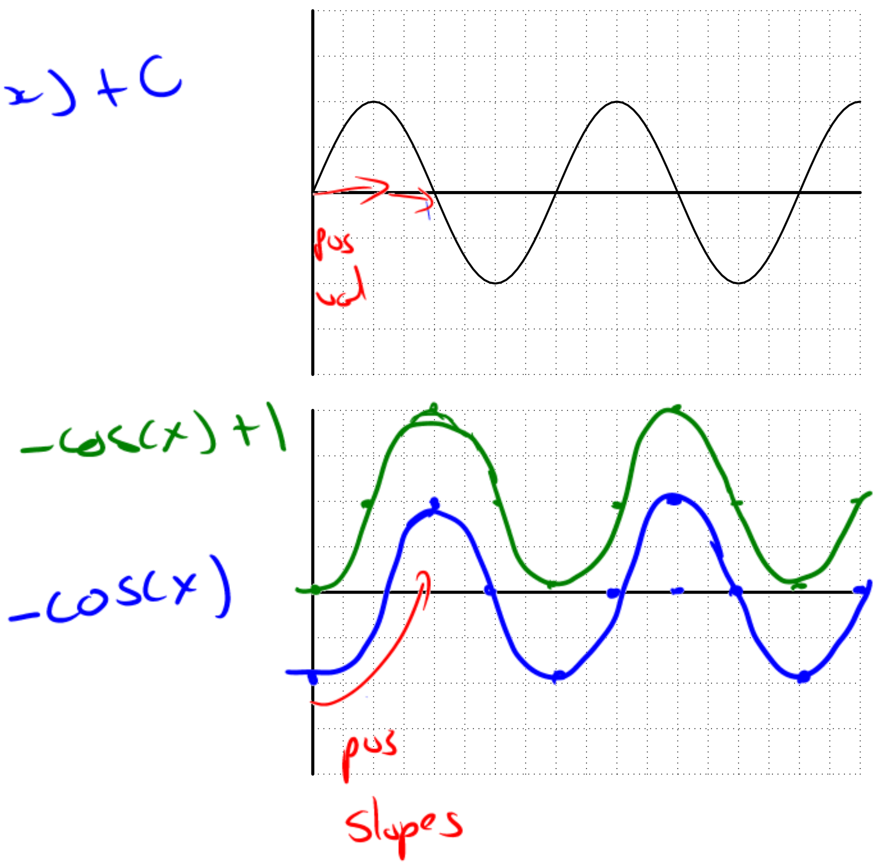
F(x)+C



Example: *Consider the graph of* $g(x) = \sin(x)$ *shown below. Sketch two possible anti-derivatives of* $g(x)$.

$f(x) = \sin(x)$

$F(x) = -\cos(x) + C$



values

slopes

The Fundamental Theorem of Calculus lets us add additional detail to the anti-derivative graph:

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Delta F$$ b/w x = a and x = b

What does this statement tell up about the graph of $F(x)$ and $f(x)$?

vertical change in F(x) (anti deriv)

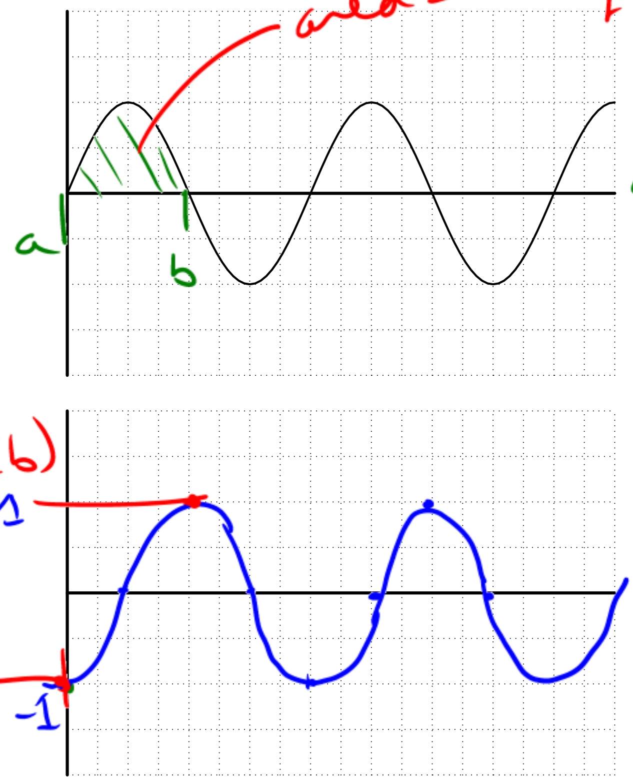
set by "area" in f(x) graph

Re-sketch the earlier anti-derivative graph of $\sin(x)$*, find the area underneath one "arch" of the sine graph.*

area = 2 sq. units

$f(x) = \sin(x)$

$F(x) = -\cos(x)$



values "areas" $\int_a^b$

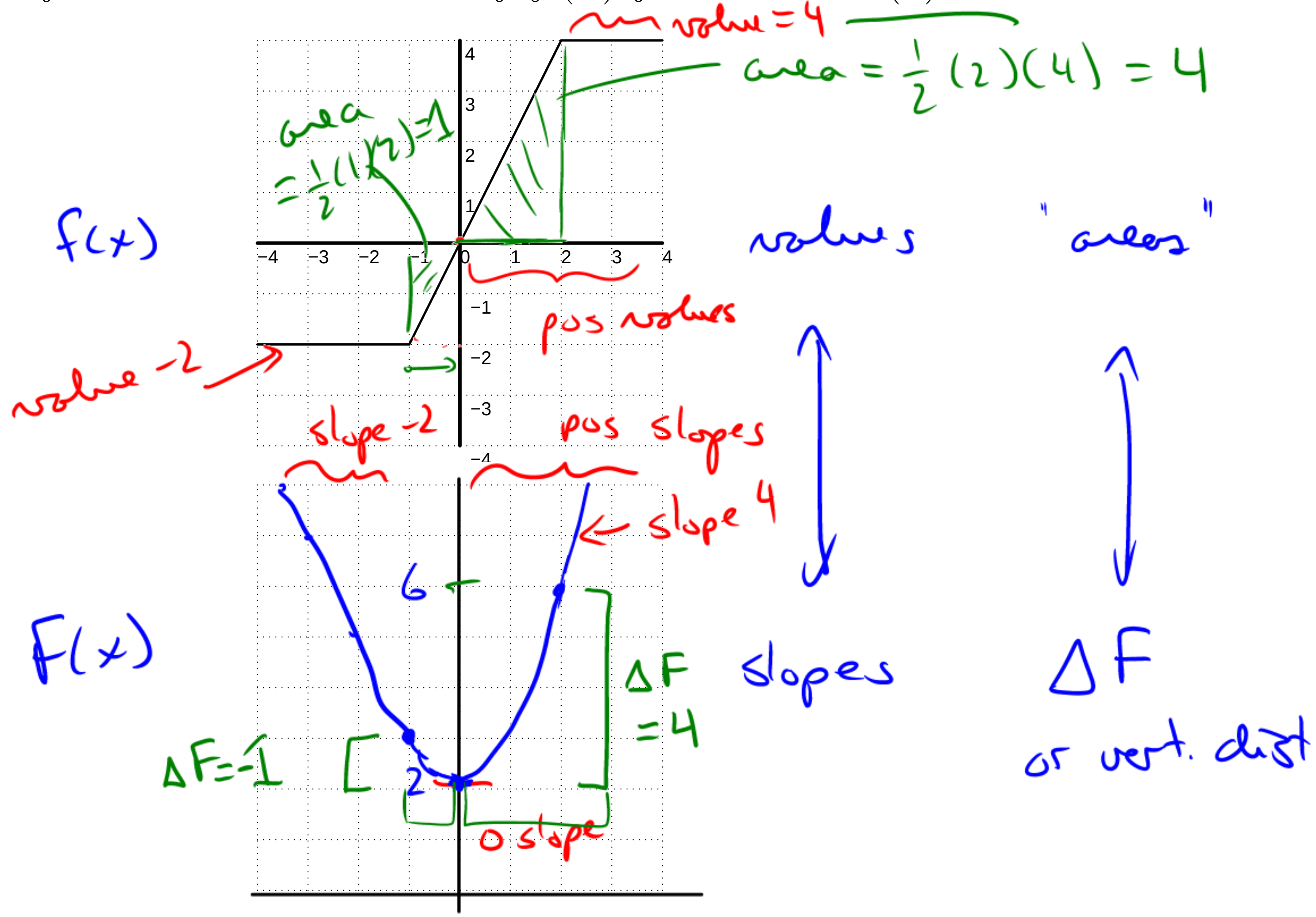
slopes vertical distances

ΔF

$F(b) - F(a) = 1 - (-1) = 2$

but $\int_a^b \underset{\sin(x)}{\underset{\shortparallel}{f(x)}}\, dx = \underbrace{F(b) - F(a)}_{\Delta F} = 2$

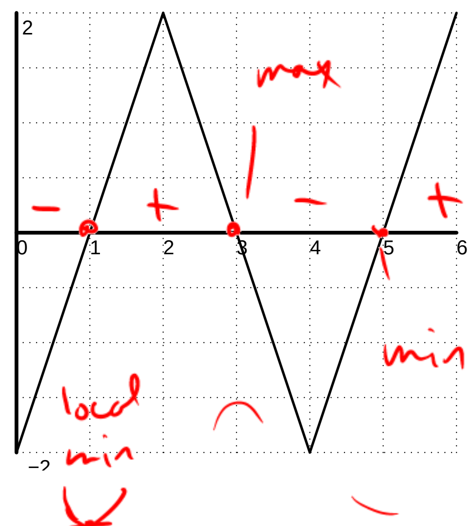
Example: *Use the area interpretation of ΔF or $F(b) - F(a)$ to construct a detailed sketch of the anti-derivative of $f(x)$ for which $F(0) = 2$.*



Example: *$f(x)$ is a continuous function, and $f(0) = 1$. The graph of $f'(x)$ is shown below.*

• crit points of $f(x)$

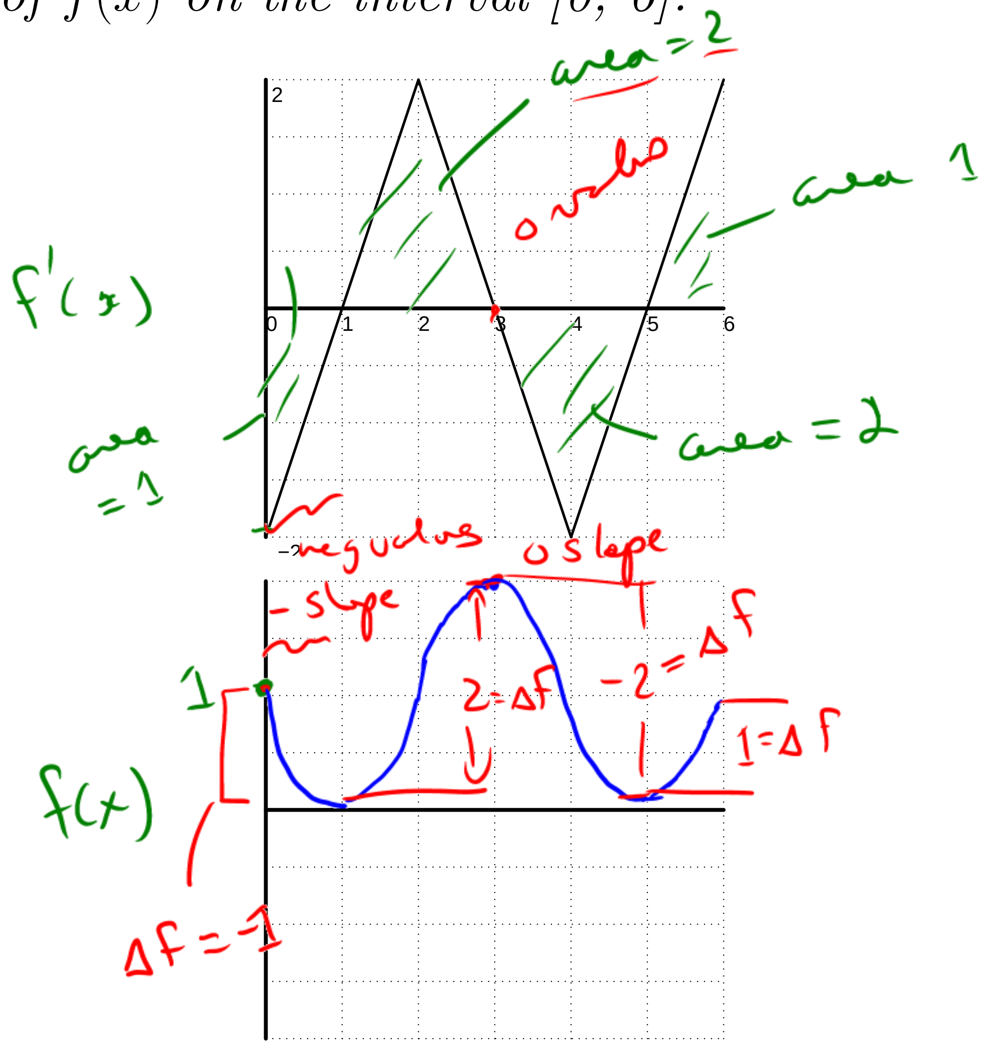
$g(x)$ $f'(x)$



$G(x)$ $f(x)$

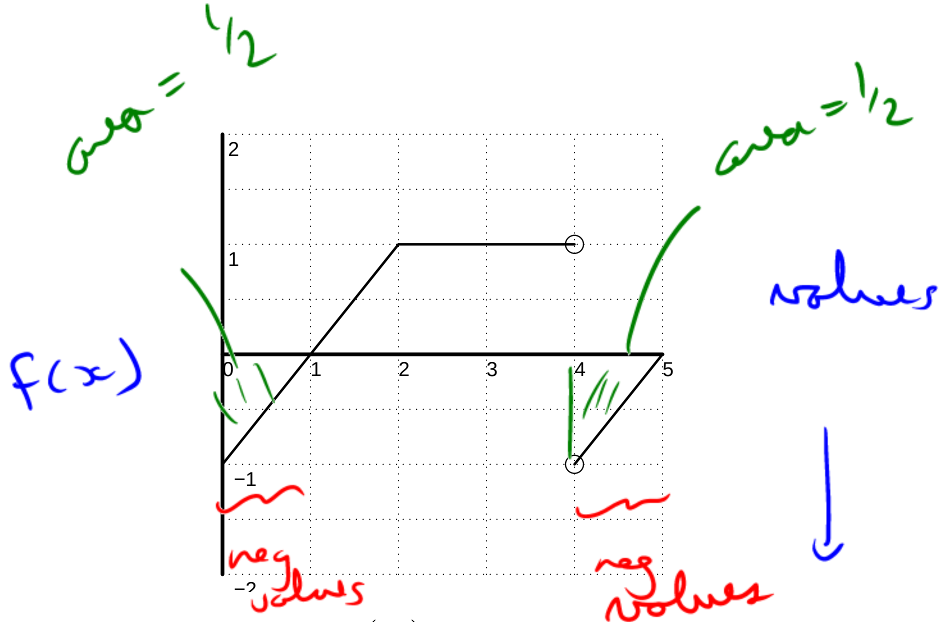
Based only on the information in the f' graph, identify the location ~~of~~ and types of all the critical points of $f(x)$ on the interval $x \in [0, 6]$.

Sketch the graph of $f(x)$ on the interval [0, 6].

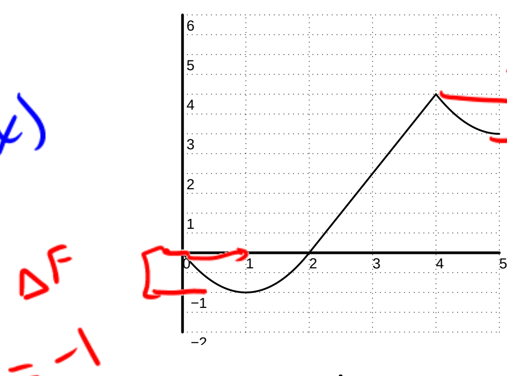


"areas" values

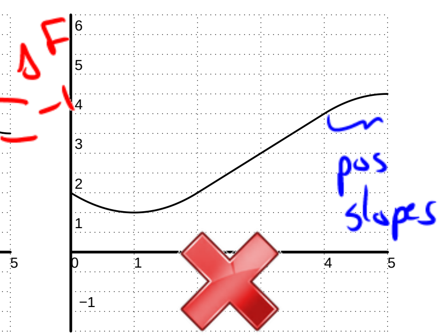
Δf slopes



Question: Given the graph of $f(x)$ above, which of the graphs below is an anti-derivative of $f(x)$?

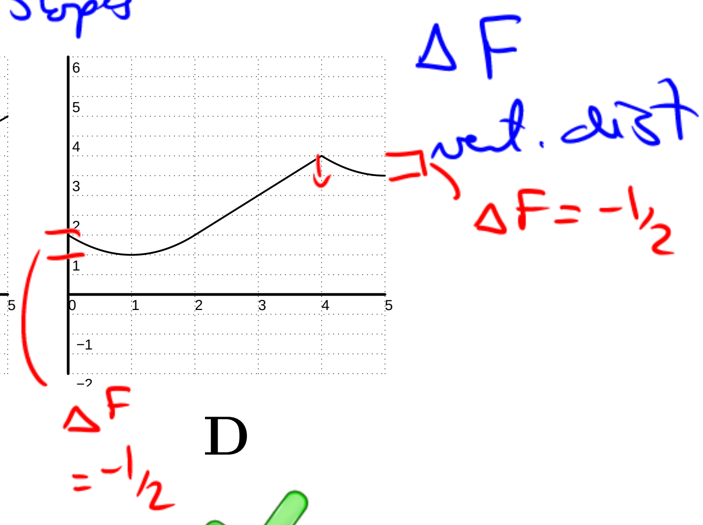


A



B

C



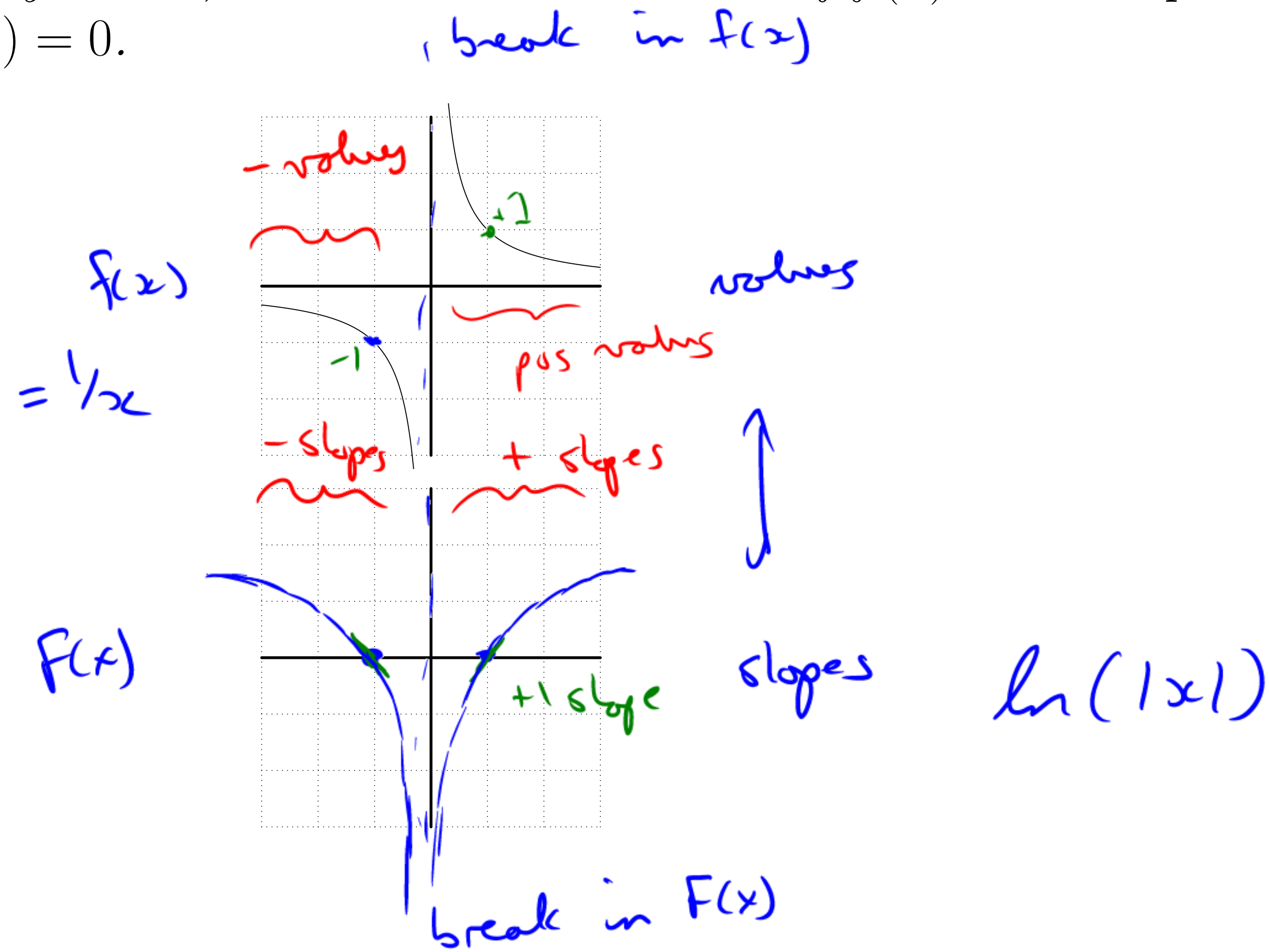
D

At the end of last week, we considered a special case to our table of anti-derivatives.

Example: *What function, differentiated, gives* $f(x) = \dfrac{1}{x}$*?*

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

Below is the graph of $f(x) = 1/x$*. Using the answer above, and our sketching techniques from today's class, sketch an anti-derivative of* $f(x)$*. Use the points* $F(1) = 0$ *and* $F(-1) = 0$*.*



Explain why we need to define $\int \frac{1}{x}\,dx = \ln(|x|) + C$*, not simply* $\ln(x) + C$.

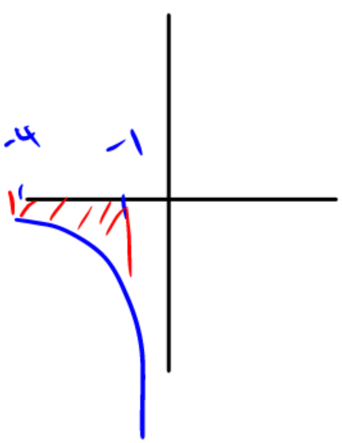
b/c $\frac{1}{x}$ is defined for all $x \neq 0$,

including neg. x values,

but $\ln(x) + C$ is only defined for pos x values

$\Rightarrow$ we needed to enlarge the domain of the anti-deriv

Sketch the 'area' implied by the integral $\int_{-4}^{-1} \frac{1}{x}\,dx$*, and find its value using the Fundamental Theorem of Calculus.*



anti-deriv

eval. $\ln(|x|)$ at $x = -1, -4$

$$\int_{-4}^{-1} \frac{1}{x}\,dx = \ln(|x|)\Big|_{-4}^{-1}$$

$$= \ln(|-1|) - \ln(|-4|)$$

$$= \ln(1) - \ln(4)$$

$$\approx 0 - 1.386 = -1.386$$

We now return to the challenge of finding a *formula* for an anti-derivative function. We saw simple cases last week, and now we will extend our methods to handle more complex integrals.

Anti-differentiation by Inspection: The Guess-and-Check Method

$$\int x^2 \, dx = \frac{x^3}{3} + C$$

d/dx

Reading: Section 7.1

Often, even if we do not see an anti-derivative immediately, we can make an educated guess and eventually arrive at the correct answer.

[See also H-H, p. 332-333]

Example: *Based on your knowledge of derivatives, what should the anti-derivative of* $\cos(3x)$, $\int \cos(3x)\, dx$, *look like?*

$$\int \cos(3x)\, dx = \frac{1}{3}\sin(3x) + C$$

$$\|$$

$$\frac{1}{\cancel{3}}\left(\cos(3x)\cdot \cancel{3}\right) \xleftarrow{d/dx} \frac{1}{3}\sin(3x)$$

d/dx

Example: *Find* $\int e^{3x-2}\,dx.$

$$\int e^{3x-2}\,dx = \frac{1}{3}e^{3x-2} + C$$

$$\frac{1}{\cancel{3}} e^{3x-2} \cdot \cancel{3}$$

$$\frac{1}{3}e^{3x-2}$$

d/dx

d/dx

$3x$, $3x-2$

Example: *Both of our previous examples had* linear *'inside' functions. Here is an integral with a* quadratic *'inside' function:*

$$\int x e^{-x^2}\, dx$$

Evaluate the integral.

$$\int x e^{-x^2}\, dx = -\frac{1}{2} e^{-x^2} + C$$

$$\frac{1}{\cancel{-}2} e^{-x^2} \cdot (\cancel{-}2x) \xleftarrow{d/dx} -\frac{1}{2} e^{-x^2} + C \quad \text{check}$$

Why was it important that there be a factor x in front of e^{-x^2} in this integral?

b/c it was the result of applying the chain rule to the "inside" $-x^2$ function

Integration by Substitution

We can formalize the guess-and-check method by defining an *intermediate variable* the represents the "inside" function.
Reading: Section 7.1

Example: *Show that* $\int x^3 \cdot \sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2} + C.$

$\not= \frac{x^4}{4} \cdot (x^4 5)^{3/2} \xrightarrow{d/dx} \not= x^3\sqrt{x^4+5}$

d/dx

check

$$\frac{d}{dx}\left(\frac{1}{6}(x^4+5)^{3/2} + C\right)$$

$$= \frac{1}{\cancel{6}}\left(\frac{3}{\cancel{2}}\right)(x^4+5)^{1/2} \cdot (\cancel{4}x^3)$$

$$= x^3\sqrt{x^4+5} =$$

✓

Relate this result to the **chain rule**.

$$\frac{d}{dx}\left(x^4+5\right)^{3/2} = \text{consts}\,\big(\underbrace{x^4+5}_{f}\big)^{1/2}\big(\underbrace{x^3}_{f'}\big)$$

f' from chain rule.

integration

or

anti-differentiation.

Now use the **method of substitution** *to evaluate* $\int x^3\sqrt{x^4+5}\,dx$

let $w = x^4+5$ ("inside" funct, w/ deriv of w "outside")

find $\frac{dw}{dx} = 4x^3$

solve for dx: $\frac{dw}{4x^3} = dx$

Rewrite integral using w's, not x's

$$\int x^3\sqrt{x^4+5}\,dx = \int \cancel{x^3}\,(w)^{1/2}\,\frac{dw}{4\cancel{x^3}} = \frac{1}{4}\int w^{1/2}\,dw$$

• simpler integral than we began with

w antideriv

$$= \frac{1}{4}\,\frac{w^{3/2}}{3/2} + C \quad \text{back to } x\text{'s} \quad = \frac{2}{4\cdot 3}\,(x^4+5)^{3/2} + C = \frac{1}{6}\,(x^4+5)^{3/2} + C$$

Steps in the Method Of Substitution

1. Select a simple function $w(x)$ that appears in the integral.
   - Typically, you will also see w' as a **factor** in the integrand as well.
2. Find $\dfrac{dw}{dx}$ by differentiating. Write it in the form $dw = \ldots\ dx$
3. Rewrite the integral using only w and dw (no x nor dx).
   - If you can now evaluate the integral, the substitution was effective.
   - If you cannot remove all the x's, or the integral became harder instead of easier, then either try a different substitution, or a different integration method.

Example: *Find* $\int \tan(x)\,dx$.

only 1 func. $w = ?$

$\frac{dw}{dx} = ?$

Rewrite before we begin integration

$$I = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$\approx dw/dx$

w

let $\boxed{w = \cos(x)}$

so $\frac{dw}{dx} = -\sin(x) \to \boxed{\frac{dw}{-\sin(x)} = dx}$

Rewrite integral w/ w's

$\sim$ evaluate anti-deriv

$$I = \int \cancel{\sin(x)} \cdot \frac{1}{w}\left(\frac{dw}{-\cancel{\sin(x)}}\right) = -\int \frac{1}{w}\,dw = -\ln(|w|) + C$$

back to x's $= -\ln(|\cos(x)|) + C$

Though it is not required unless specifically requested, it can be reassuring to check the answer.

Verify that the anti-derivative you found is correct.

check: is $\frac{d}{dx}\left[-\ln(|\cos(x)|)+C\right] = \frac{\sin(x)}{\cos(x)} = \tan(x)$?

$= + \frac{1}{\cos(x)} \cdot (+\sin(x))$

$= \frac{\sin(x)}{\cos(x)} = \tan(x)$ ✓

$\frac{d}{dx}\ln(|x|)$

$= 1/x$

no abs

so $\int \tan(x)\,dx = -\ln|\cos(x)| + C$

$w = x^4 - 3$

Example: *Find* $\int x^3 e^{x^4-3}\,dx. = I$

let $w = x^4 - 3$

so $\frac{dw}{dx} = 4x^3 \rightarrow \frac{dw}{4x^3} = dx$

simpler than original

so $I = \int \cancel{x^3} e^w \left(\frac{dw}{4\cancel{x^3}}\right) = \frac{1}{4}\int e^w \cdot dw$

d/dx ✓

anti-deriv

$= \frac{1}{4} e^w + C$

back to x's

$= \frac{1}{4} e^{x^4-3} + C$ ✓

Example: *For the integral,*

$\frac{d}{dx}\, e^{x} + e^{-x}$

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} dx$$

$\frac{d}{dx} = e^{x} - e^{-x}$

both $w = e^x - e^{-x}$ and $w = e^x + e^{-x}$ are seemingly reasonable substitutions.

Question: Which substitution will change the integral into a simpler form?

1. $w = e^x - e^{-x}$

2. $w = e^x + e^{-x}$

Compare both substitutions in practice.

$$I = \int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} dx$$

with $w = e^x - e^{-x}$

so $\frac{dw}{dx} = e^x + e^{-x}$

or $\frac{dw}{(e^x + e^{-x})} = dx$

so

$$I = \int \frac{w}{(e^x+e^{-x})^2} \left(\frac{dw}{(e^x+e^{-x})} \right)$$

$$= \int \frac{w}{(e^x+e^{-x})^3} dw$$

still a mix of w's, x's: can't integrate

⇒ this w likely a poor choice.

with $w = e^x + e^{-x}$

so $\frac{dw}{dx} = e^x - e^{-x}$ or $\frac{dw}{e^x - e^{-x}} = dx$

so $I = \int \frac{\cancel{e^x - e^{-x}}}{w^2} \left(\frac{dw}{\cancel{e^x - e^{-x}}} \right)$

$$= \int \frac{1}{w^2} dw = \int w^{-2} dw$$

simpler than original ✓

antideriv

$$= \frac{w^{-1}}{-1} + C$$

back to x's

$$= \frac{-1}{(e^x + e^{-x})} + C$$

Example: *Find* $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}dx. = I$

$w = \sin(x)$ ✗ (no x (deriv) in integral)

$w = 1 + \cos^2(x) \rightarrow \frac{dw}{dx} = 2\cos(x)\cdot(-\sin(x))$ no nice cancellation

Try $w = \cos(x)$

so $\frac{dw}{dx} = -\sin(x) \Rightarrow \frac{dw}{-\sin(x)} = dx$

easier than original?

so $I = \displaystyle\int \frac{\cancel{\sin(x)}}{1+w^2}\left(\frac{dw}{-\cancel{\sin(x)}}\right) = -\int \frac{1}{1+w^2}\,dw$

recall

$\left(\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}\right)$

$= -\arctan(w) + C$

$= -\arctan(\cos(x)) + C$ ✓

Using the Method of Substitution for Definite Integrals

If we are asked to evaluate a **definite** integral such as

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx \quad ,$$

where a substitution will ease the integration, we have two methods for handling the limits of integration ($x = 0$ and $x = \pi/2$).

a) When we make our substitution, convert both the *variables* x and the *limits* (in x) to the new variable; or

b) do the integration keeping the limits explicitly in terms of x, writing the final integral back in terms of the original x variable as well, and *then* evaluating.

write limits as w values

Example: *Use method a) to evaluate the integral*

$$I = \int_0^{\pi/2} \frac{\sin x}{1+\cos x}dx$$

let $\boxed{w = 1+\cos(x)}$

$x = 0 \Rightarrow w = 1 + \cos(0) = 2$

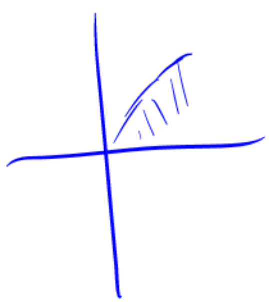
so $\frac{dw}{dx} = -\sin(x)$ or $\boxed{\frac{dw}{-\sin(x)} = dx}$

$x = \pi/2 \Rightarrow w = 1 + \cos(\pi/2) = 1$

$$I = \int_{w=2}^{w=1} \frac{\cancel{\sin(x)}}{w}\left(\frac{dw}{-\cancel{\sin(x)}}\right) = -\int_{w=2}^{w=1} \frac{1}{w}\,dw = -\ln|w|\Big|_{w=2}^{w=1}$$

$$= -(\ln|1| - \ln|2|)$$

$$= -(\ln(1) - \ln(2)) \cong +0.693$$



keep the limits in x explicitly

Example: *Use method b) method to evaluate*

$$I = \int_{x=9}^{x=64} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx \;.$$

let $\boxed{w = 1+\sqrt{x}}$

so $\dfrac{dw}{dx} = \dfrac{1}{2}\dfrac{1}{\sqrt{x}}$ $\left(\text{or } \tfrac{1}{2}x^{-1/2}\right) \rightarrow \boxed{2\sqrt{x}\,dw = dx}$

so $I = \displaystyle\int_{x=9}^{x=64} \frac{\sqrt{w}}{\cancel{\sqrt{x}}}\Big(\underbrace{2\cancel{\sqrt{x}}\,dw}_{dx}\Big) = 2\int_{x=9}^{x=64} \underset{w^{1/2}}{\sqrt{w}}\,dw = 2\,\frac{w^{3/2}}{3/2}\Bigg|_{x=9}^{x=64}$

back to x's

$= \dfrac{4}{3}\left(1+\sqrt{x}\right)^{3/2}\Bigg|_{x=9}^{x=64} = \dfrac{4}{3}\left((1+8)^{3/2} - (1+3)^{3/2}\right)$

now match ✓

$= \tfrac{4}{3}\left(27-8\right) = \dfrac{4\cdot 19}{3} = \dfrac{76}{3}$

both w and (w') present

Non-Obvious Substitution Integrals

Sometimes a substitution will still simplify the integral, even if you don't see an obvious cue of "function and its derivative" in the integrand.

Example: *Find*

$$I = \int \frac{1}{\sqrt{x}+1}\,dx \quad . \qquad = ??$$

$= x^{1/2}$

Try $\boxed{w = \sqrt{x} + 1} \Rightarrow \sqrt{x} = w - 1$

so $\dfrac{dw}{dx} = \dfrac{1}{2} x^{-1/2}$ or $2\sqrt{x}\,dw = \underline{\underline{dx}}$

mix of w's, x's

so $I = \int \dfrac{1}{w}\left(2\underline{\underline{\sqrt{x}}}\,dw\right)$

$= \int \dfrac{1}{w}(2)(w-1)\,dw$

separate numerator

$= 2\int \dfrac{w-1}{w}\,dw = 2\left[\int\left(1 - \dfrac{1}{w}\right)dw\right]$

$$I = 2\int \left(1 - \frac{1}{w}\right) dw$$

$$= 2\left(w - \ln(|w|)\right) +$$

back to x's

$$= 2\left[\left(\sqrt{x}+1\right) - \ln\left|\sqrt{x}+1\right|\right] + C$$

$$\int \frac{1}{\sqrt{x}+1}\, dx =$$