Unit #18 : Level Curves, Partial Derivatives

Goals:

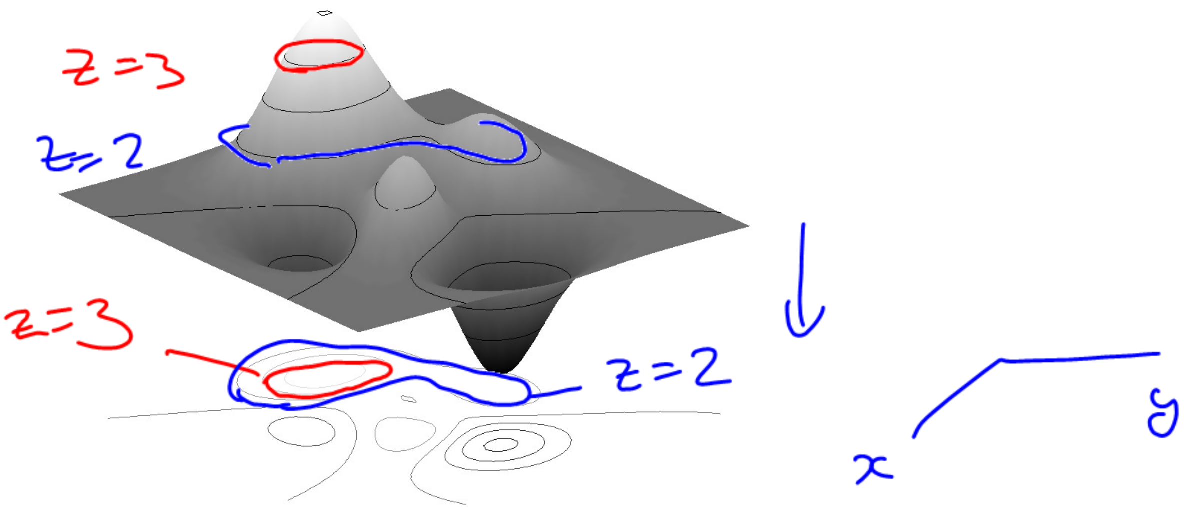
- To learn how to use and interpret contour diagrams as a way of visualizing functions of two variables.
- To study linear functions of two variables.
- To introduce the partial derivative.

Picturing $f(x, y)$: Contour Diagrams (Level Curves)

We saw earlier how to sketch surfaces in three dimensions. However, this is not always easy to do, or to interpret. A contour diagram is a second option for picturing a function of two variables.

Suppose a function $h(x, y)$ gives the height above sea level at the point (x, y) on a map. Then, the graph of h would resemble the actual landscape.

Suppose the function h looks like this:



Then, the **contour diagram** of the function h is a picture in the (x, y)–plane showing the **contours**, or **level curves**, connecting all the points at which h has the same value. Thus, the equation

set of x,y points

$$h(x, y) = \underline{\underline{100}}$$

height equals a constant

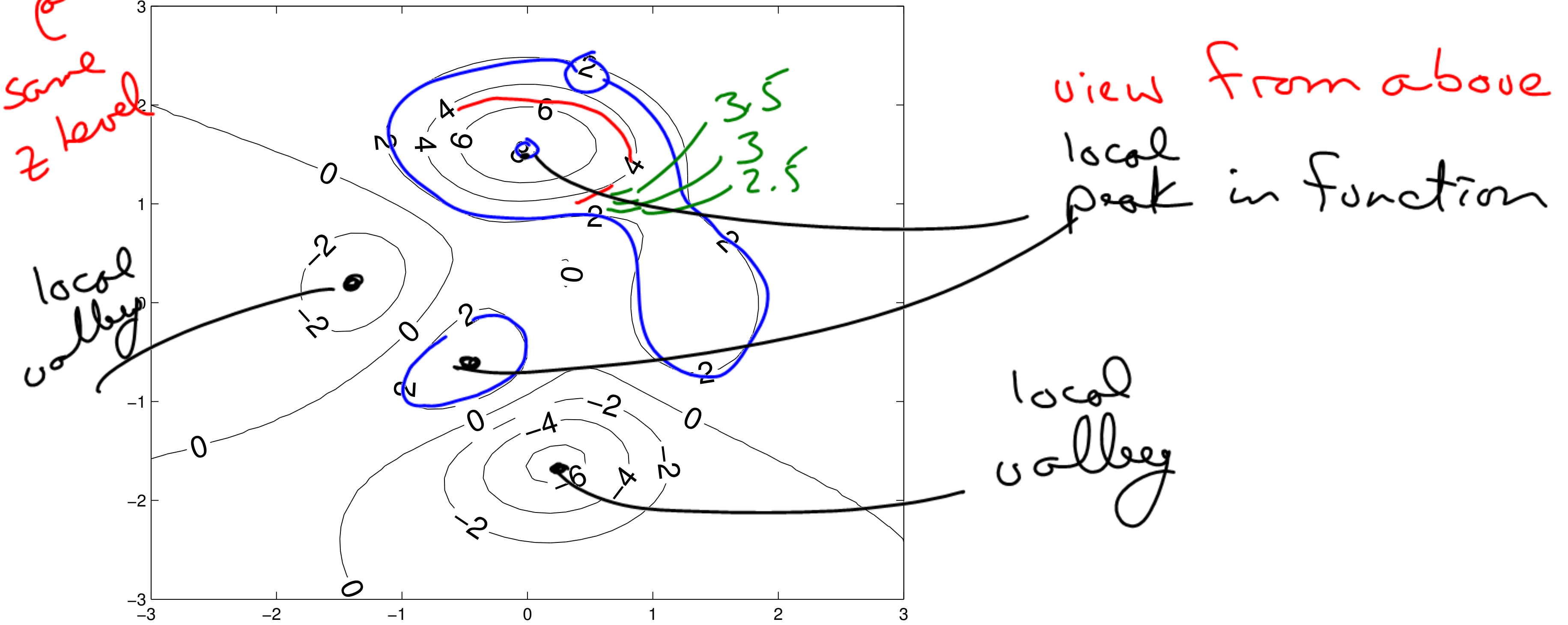
gives all the points where the function value is 100.

h(x, y) = 8

Together they usually constitute a curve or a set of curves called the contour or level curve for that value. In principle, there is a contour through every point. In practice, just a few of them are shown.

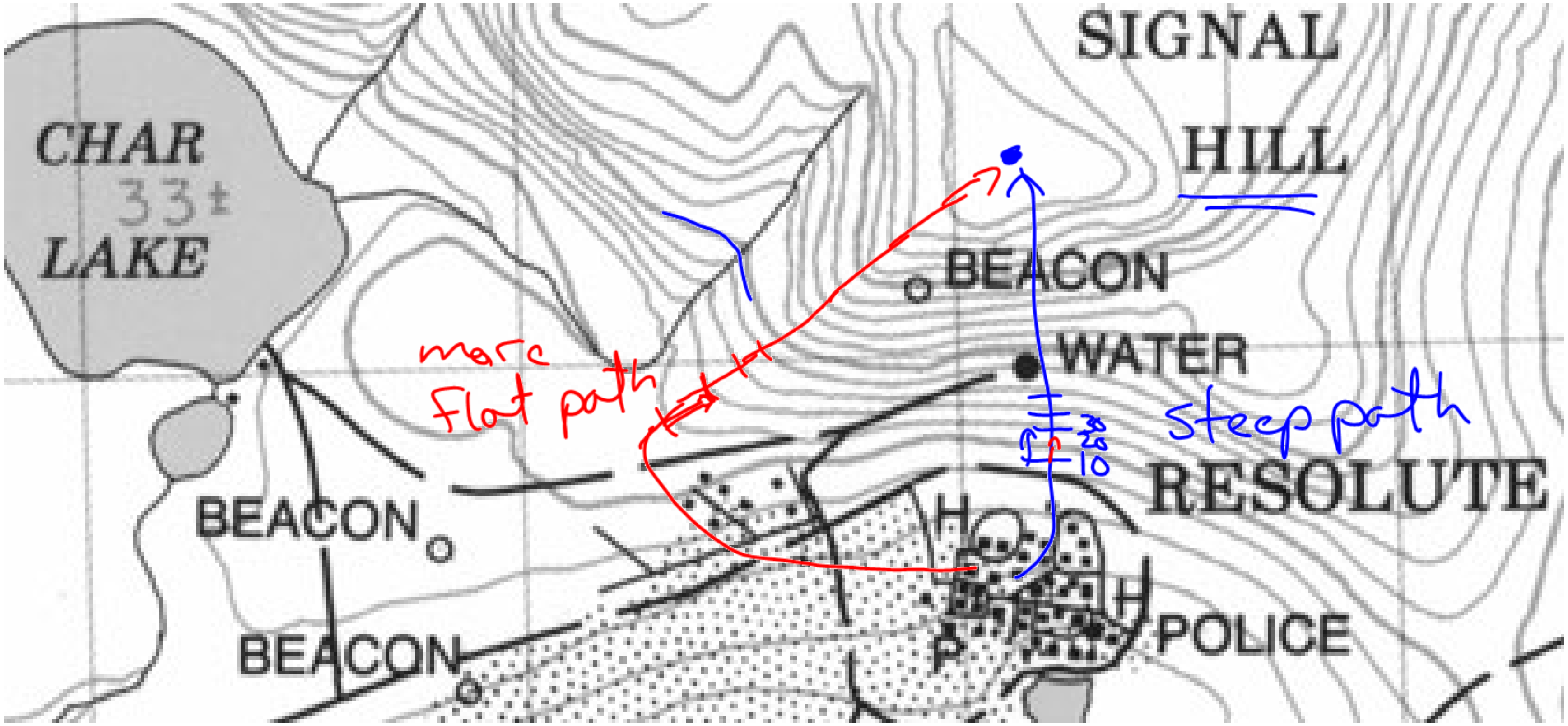
all @ same z level

The following is the contour diagram for the earlier surface.



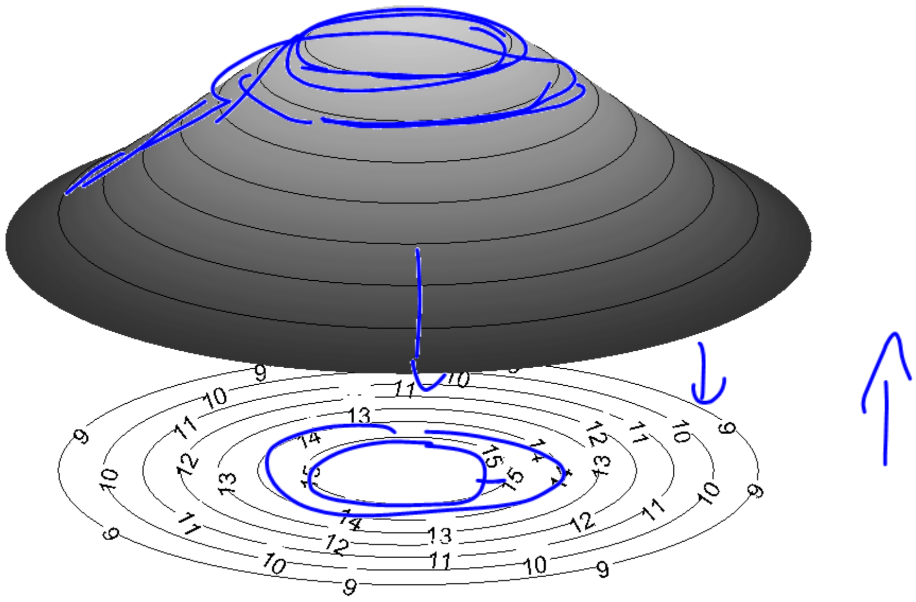
Indicate the location of the peaks and pits/valleys on the contour diagram.

Topographic maps are also contour maps.



Identify first a steep path, and then a more flat path, from the town up to Signal hill.

In principle, the contour diagram and the graph can each be reconstructed from the other. Here is a picture illustrating this:



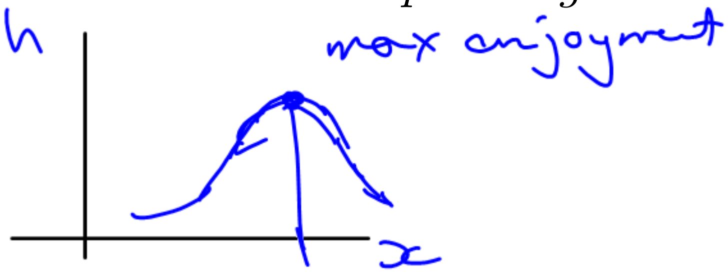
As shown above, the contour $f(x, y) = k$ is obtained by intersecting the graph of f with the horizontal plane, $z = k$, and then dropping (or raising) the resulting curve to the (x, y)–plane. The graph is obtained by raising (or dropping) the contour $f(x, y) = k$ to the level $z = k$.

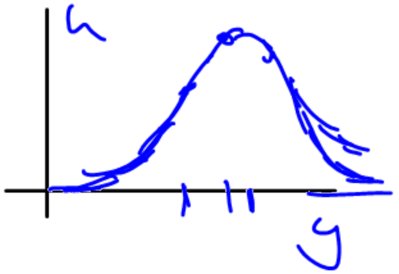
Interpreting Contour Diagrams

Match each of the following functions to their corresponding contour diagram.

(1) $h(x, y)$ is the degree of pleasure you get from a cup of coffee when

- x is the temperature, and
- y is the amount of ground coffee used to brew it.





(2) $f(x, y)$ is the number of TV sets sold when

- x is the price per TV set, and
- y is the amount of money spent weekly on advertising.

(3) $g(x, y)$ is the amount of gas per week sold by a gas station when

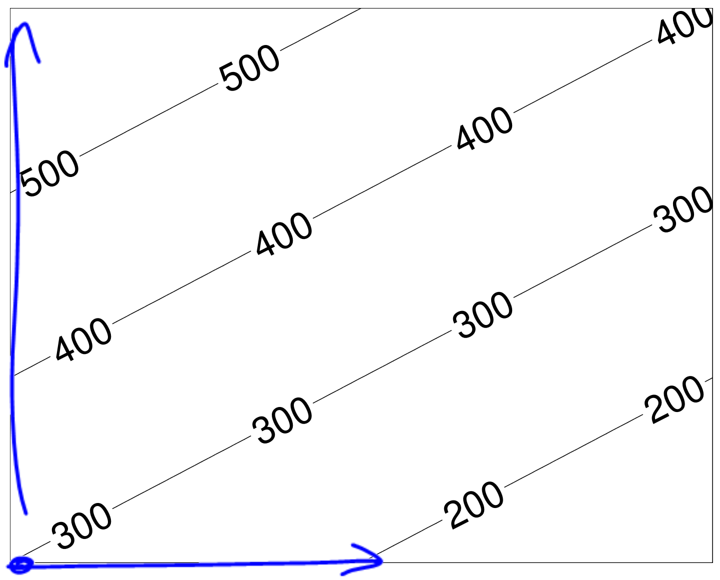
- x is the amount spent on bonus gifts to customers, and
- y is the price charged by a nearby competitor.

incr x ⇒ incr in g

incr y ⇒ incr in g

incr x → expect to sell fewer → decr in $f(x,y)$

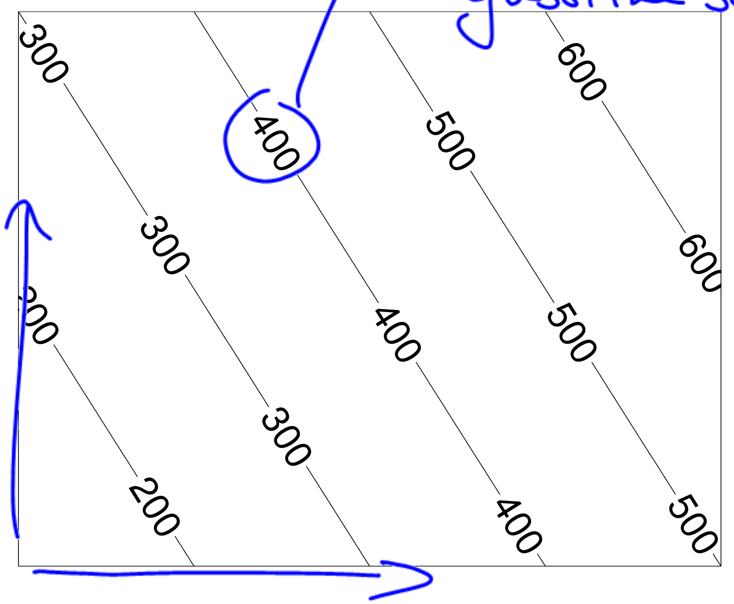
incr y → " " " more TVs → incr in $f(x,y)$



y amt spent on ads

x TV price

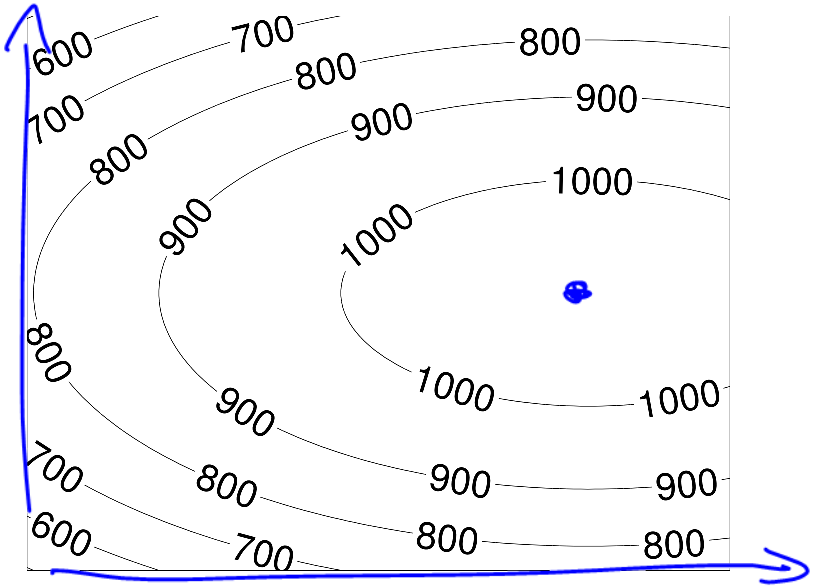
f(x,y) = # TVs sold



y competitors price

g(x,y) amt of gasoline sold

x bonus gifts



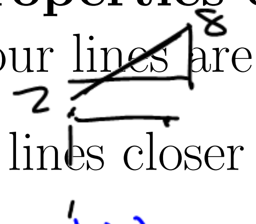
y amt of coffee

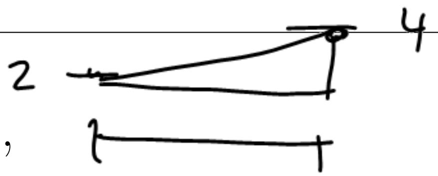
h(x,y) enjoyment of coffee

x temp

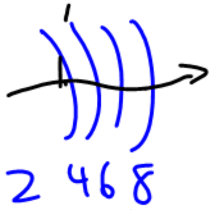
Useful Properties of Contour Plots

If the contour lines are evenly spaced in their z values,



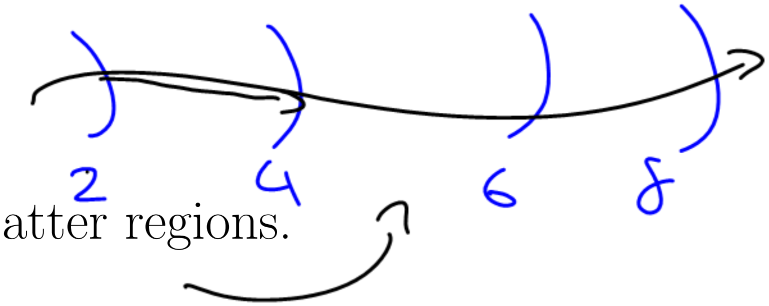


- contour lines closer together indicate more rapid change/steeper slopes.

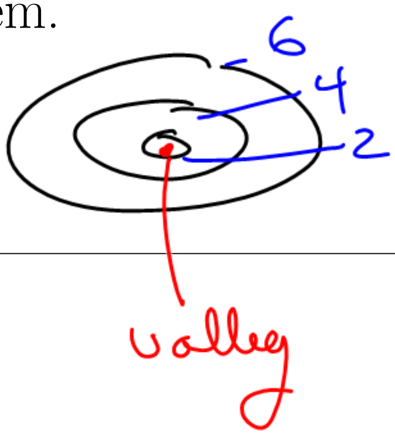


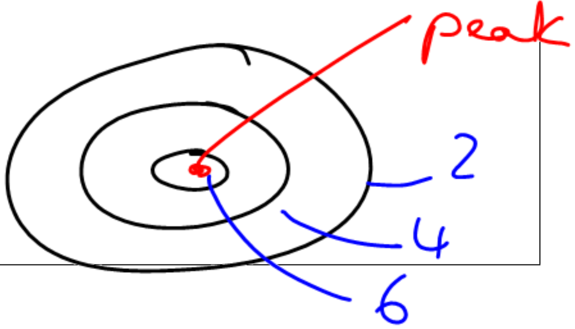
uphill

quicker / more steeply



- contour lines further apart indicate flatter regions.
- peaks and valleys look the same; only the values of the contours let you distinguish them.

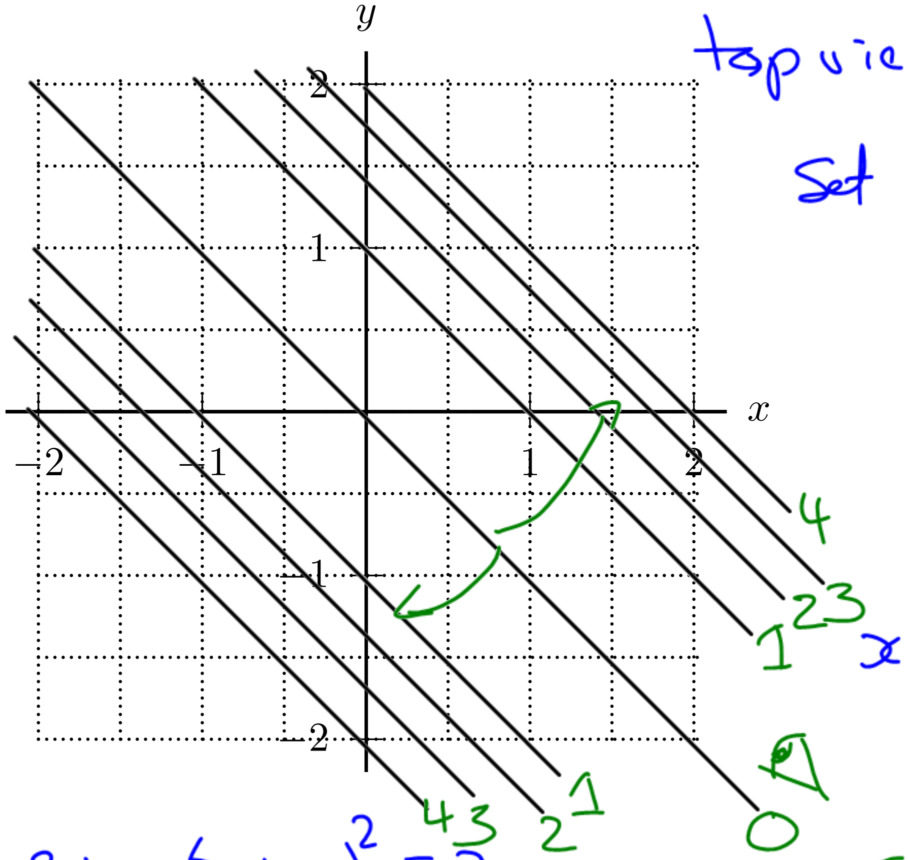




$z=0 \rightarrow (x+y)^2=0 \rightarrow x+y=0 \quad y=-x$

Example

Draw the contours of $f(x,y)=(x+y)^2$ for the values $f=1$, 2, 3, and 4.



top view

Set $\cancel{f(x,y)} = \boxed{1} - z$

$(x+y)^2$

solve for x,y

points/curves

curves

$(x+y) = \pm 1$

$x+y=1$ or $x+y=-1$

$y=1-x$ $\quad$ $y=-1-x$

straight lines w/ slope −1, intercepts 1, −1

$z=2: \quad (x+y)^2=2$

$x+y = \pm\sqrt{2}$

$y=\sqrt{2}-x$ or $y=-\sqrt{2}-x$

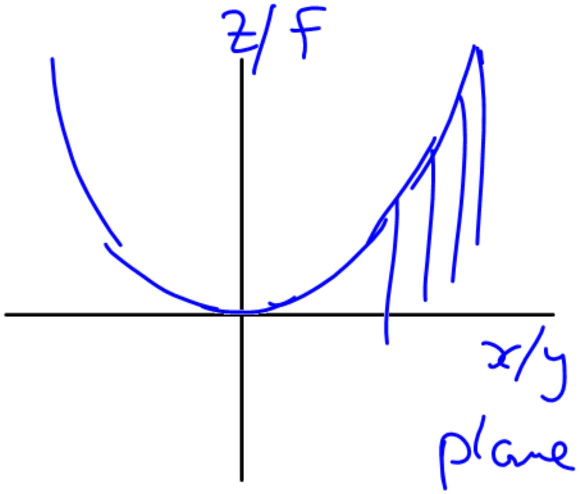
≈ -1.4

$z=3 \rightarrow y=+\sqrt{3}-x, \; y=-\sqrt{3}-x$

$z=4 \rightarrow y=+2-x, \; y=-2-x$

Give a verbal description of the surface defined by $f(x, y) = (x + y)^2$.

side view



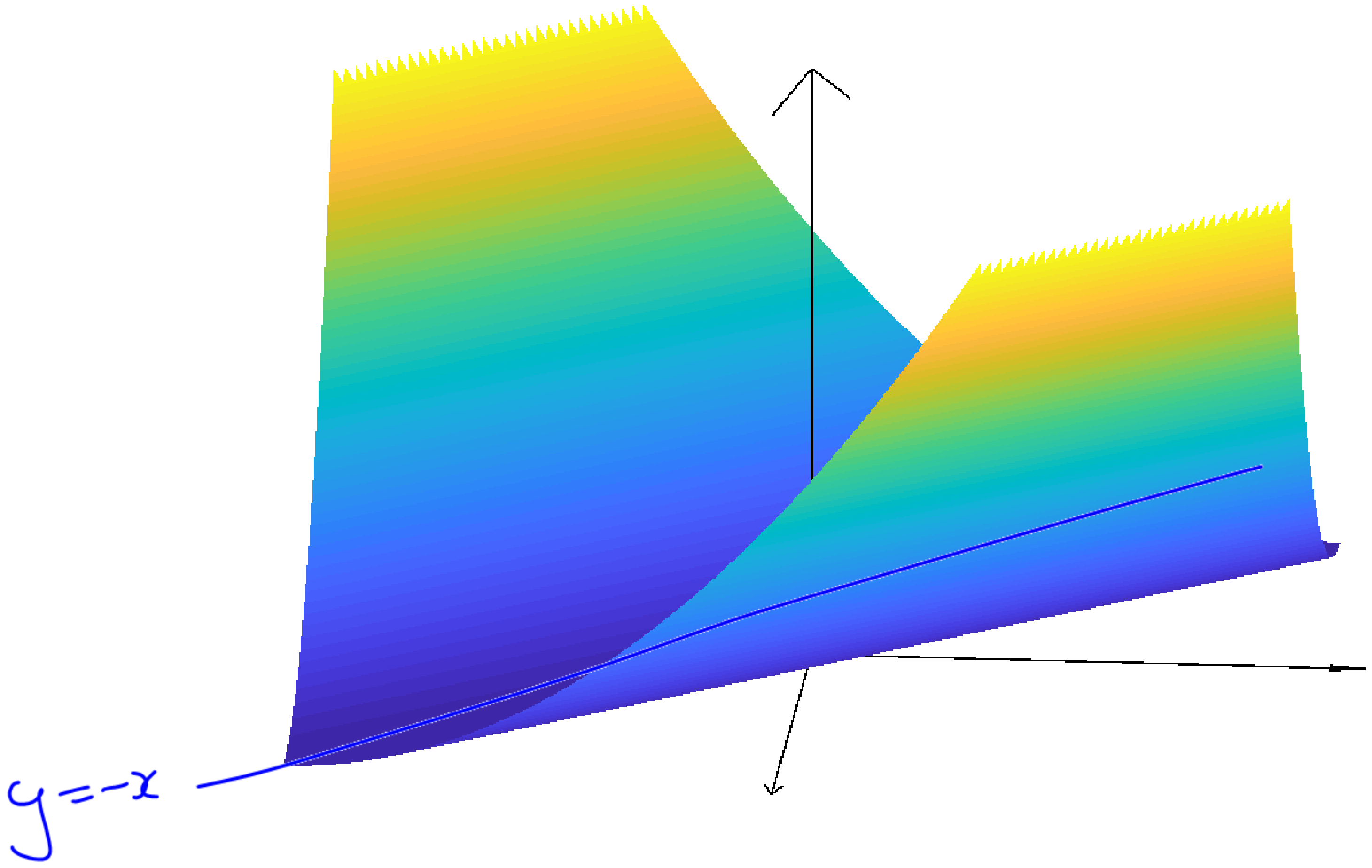
parabolic

trough

aligned on

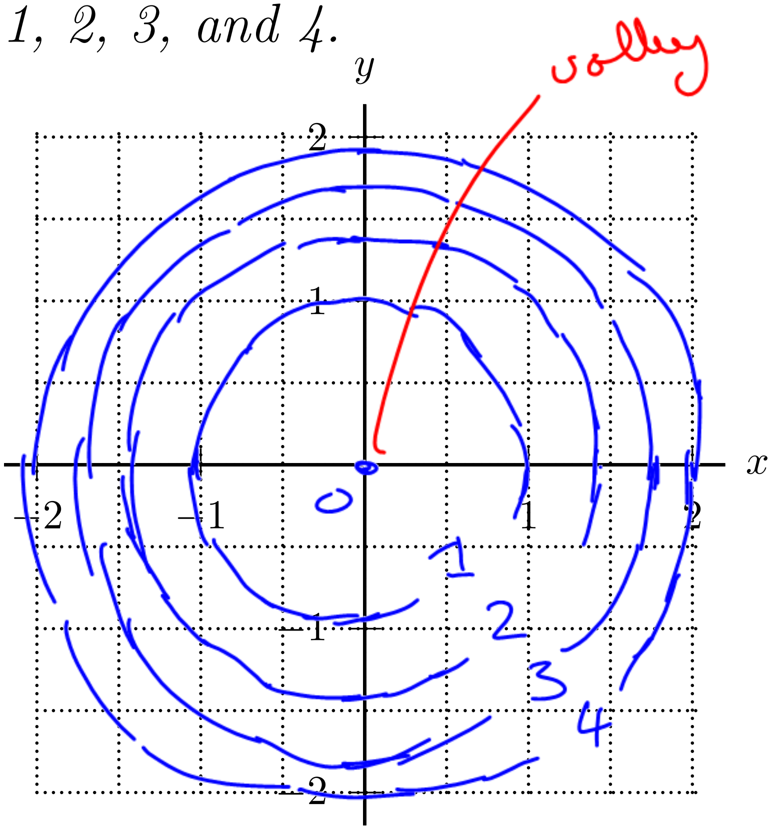
$y=-x$

Here is the function $f(x, y) = (x + y)^2$ plotted as a surface.



$x^2+y^2=r^2$

Example: *Draw the contours of* $f(x,y)=x^2+y^2$ *for the values 1, 2, 3, and 4.*



$z=1 \to \underbrace{x^2+y^2}=1$

recognize as a circle

- centered @ origin

- radius 1

$z=2 \to x^2+y^2=2=r^2$

so radius is $\sqrt{2}$

≈ 1.4

$z=3 \to x^2+y^2=3=r^2$ so

radius of $\sqrt{3}$

$z=4 \to x^2+y^2=4$ -> radius of 2

From the contour diagram, is the value of f at $\underbrace{(0,0)}$ a local minimum or maximum?

local minimum for f(x, y)

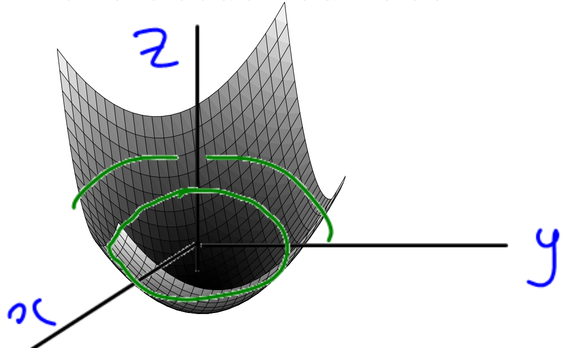
f(x, y) always increases as move away from (0,0)

Is the surface becoming more or less steep as you move away from the origin?

contours closer together as we move away from (0,0)

→ surface gets steeper

Try to associate how the lines on the contour diagram could help you to imagine the actual surface:



Linear Functions of Two Variables

z axis intecept

A function of two variables is **linear** if its formula has the form

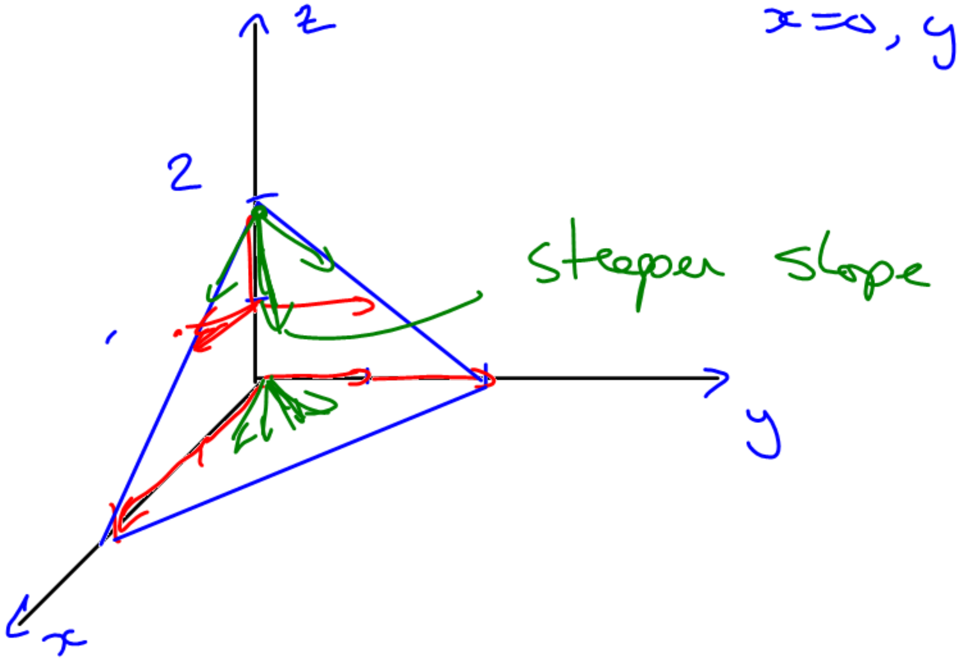
$$f(x,y) = c + mx + ny.$$

const coeff

The textbook shows that m and n can be interpreted as **slopes in the x-direction and the y-direction**, respectively, and that c **is the z-intercept**.

Consider the plane $z = 2 - x - y$. This plane has slope -1 in both the x- and the y-directions. Is there any direction in which it has a steeper slope?
It may help to experiment by holding up a book or other flat object.

$x=0, y=0 \rightarrow z=2$



Linear Example

Given that the following is a table of values for a linear function, f, complete the table.

Δx Δx Δx

x

y	2	3	4	5
1.1	2	9.5	17	24.5
1.2	4	11.5	19	26.5
1.3	6	13.5	21	28.5
1.4	8	15.5	23	30.5

Δz

Δz

$\Delta y = 0.1$

$+1.5$

$\Delta z = 7.5$

1.5 10 9.5

like point/slope formula

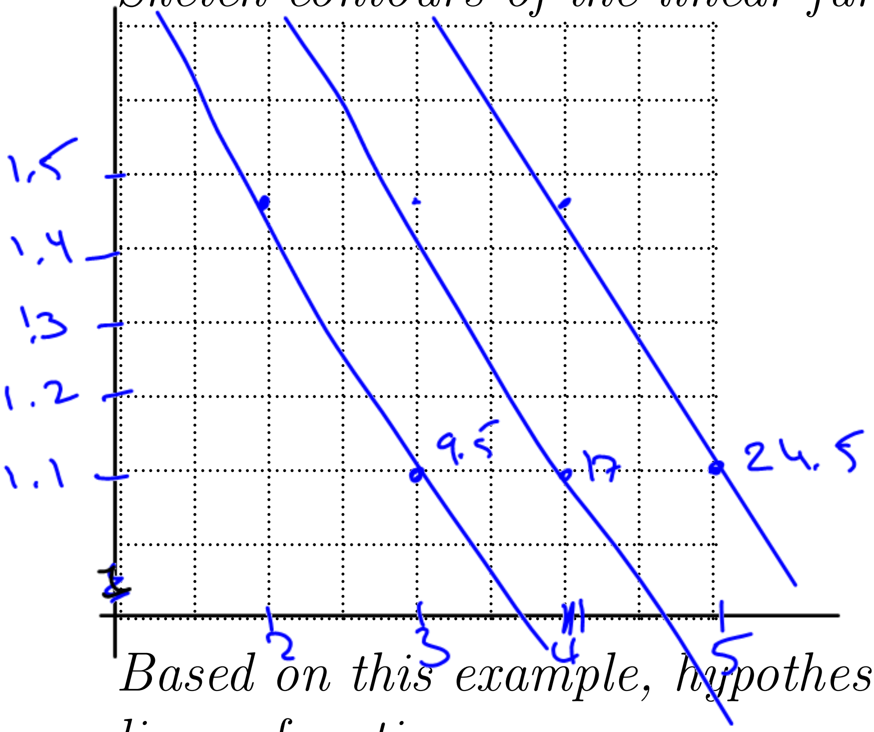
Give a formula for the function $f(x, y)$ in the preceding example.

$$x \text{ slope} = \frac{\Delta z}{\Delta x} = \frac{7.5}{1} = 7.5$$

$$\rightarrow f(x,y) = 2 + 7.5(x-2) + 20(y-1.1)$$

$$y \text{ slope} = \frac{\Delta z}{\Delta y} = \frac{2}{0.1} = 20$$

Sketch contours of the linear function f.



contour

$$\underset{z_0}{\cancel{z}} = c + nx + my$$

$$\hookrightarrow my = z_0 - c - nx$$

$$y = \frac{z_0 - c - nx}{m}$$

linear function in x, y

Based on this example, hypothesize properties of the contour diagrams for all linear functions.

- contours are all straight lines
- x/y plane slopes will be the same / parallel
- evenly spaced

Partial Derivatives

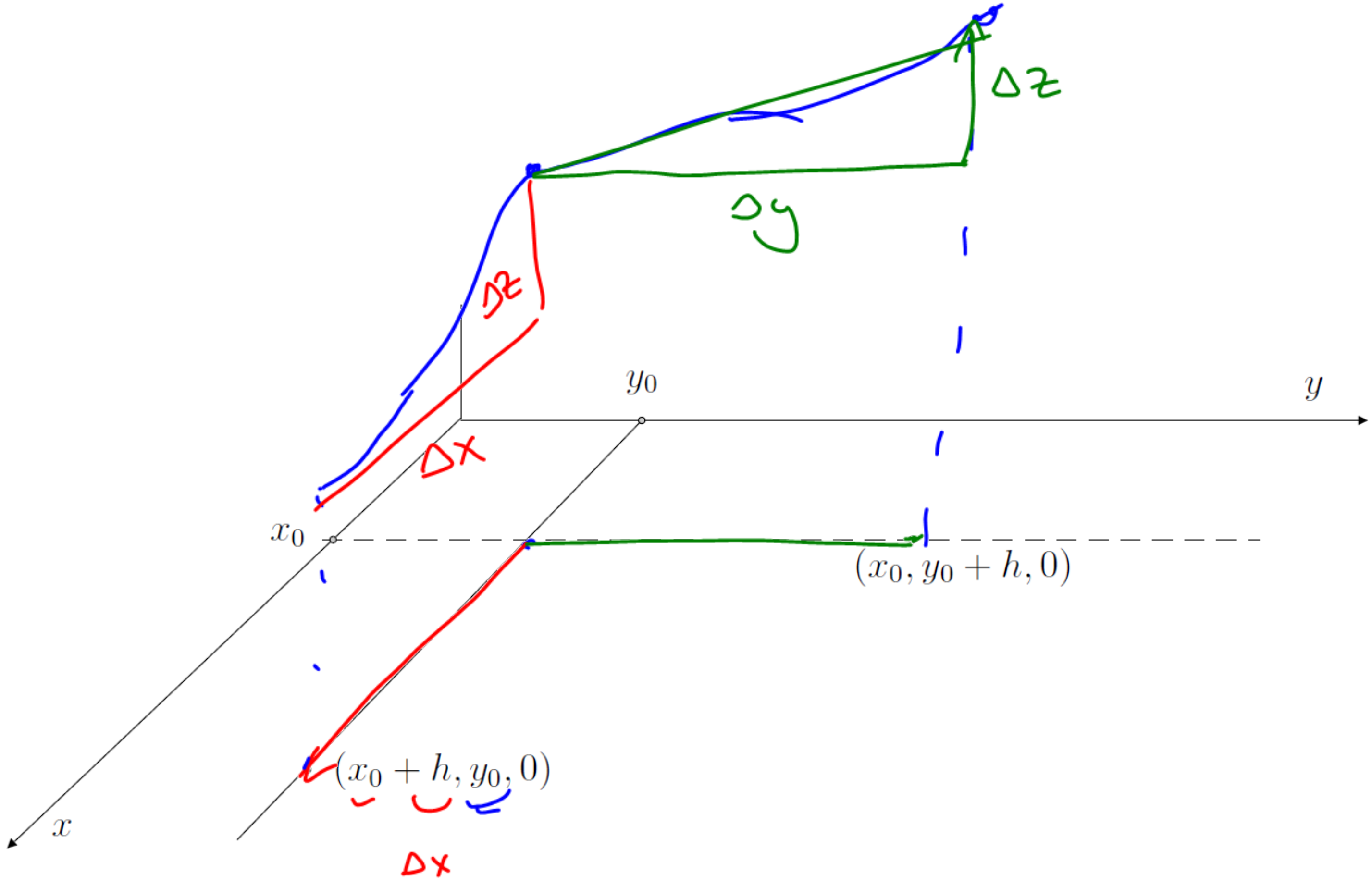
Just as $\dfrac{df}{dx}$ is the rate of change of $f(x)$ when x is changed, so the derivatives of $f(x, y)$ are the rates of change of the function value when **one** of the variables is changed. Since there are two variables to choose from, there are two derivatives, one to describe what happens when you change x **only** and one to describe what happens when you change y **only**. Because either derivative by itself describes the behaviour of the function only **partly**, they are called **partial derivatives**.

surface

If we look at a graph of $z = f(x, y)$, partial derivatives tell us how the height of the graph, z, is changing as the point $(x, y, 0)$ moves along a line parallel to the x-axis or parallel to the y-axis.

$(x_0, y_0, 0) \longrightarrow (x_0 + h, y_0, 0)$ (along solid line parallel to the x-axis).

$(x_0, y_0, 0) \longrightarrow (x_0, y_0 + h, 0)$ (along dotted line parallel to the y-axis).



Definition of Partial Derivatives

"die" →

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \approx \frac{\Delta f}{\Delta x}$$

$$= f_{\boxed{x}}(x_0, y_0)$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \approx \frac{\Delta f}{\Delta y}$$

$$= \boxed{f_y}(x_0, y_0)$$

As a shorthand,

- $f_x(x_0, y_0)$ also means the same as $\frac{\partial f}{\partial x}(x_0, y_0)$, and
- $f_y(x_0, y_0)$ also means the same as $\frac{\partial f}{\partial y}(x_0, y_0)$.

There is no $\underline{\underline{\cancel{f'(x, y)}}}$ notation possible when working with multivariate functions.

To actually calculate $\dfrac{\partial f}{\partial x}$ when we have a formula for $f(x, y)$, we imagine that y is fixed, then we have a function of only ONE variable and we take its derivative in the usual way. To calculate $\dfrac{\partial f}{\partial y}$, we imagine that x is fixed and y is not.

fix / make constant

Example: *Consider* $f(x,y) = x^2y + \sin(xy)$.

Write a new single-variable function: $g(\underset{\uparrow}{x}) = f(x, \overset{\downarrow}{\underline{y_0}})$

one variable

$$g(x) = x^2 \cdot \underset{\uparrow}{y_0} + \sin(x \cdot \underset{\uparrow}{y_0})$$

constant

Find $\dfrac{dg}{dx}$.

$$\frac{dg}{dx} = 2x \cdot y_0 + \cos(x \cdot y_0) \cdot (1 \cdot y_0)$$

$$= 2xy_0 + y_0 \cos(x \cdot y_0)$$

$$\Rightarrow \quad f_x \text{ or } \boxed{\frac{\partial f}{\partial x} = 2xy + y\cos(xy)}$$

More practically, when we are looking for $\frac{\partial f}{\partial x}(x, y)$ — a formula in terms of x and y — we do not actually replace y by y_0, but simply *think of it* as a constant.

$$f(x, y) = x^2 y + \sin(xy)$$

x, y are unrelated variables

Find $\frac{\partial f}{\underline{\partial x}}(x, y)$.

$$\frac{\partial f}{\partial x} = 2x \cdot y + \cos(xy) \cdot \frac{\partial}{\partial x}(xy)$$

$$\frac{\partial f}{\partial x} = 2xy + \cos(xy) \cdot y$$

Find $\frac{\partial f}{\partial y}(x, y)$.

$$\frac{\partial f}{\partial y} = x^2 \cdot 1 + \cos(xy) \cdot x$$

$$f(x,y) = \overset{\text{const}}{\overset{\downarrow}{x^2}}\,\boxed{y} + \sin(x\boxed{y})$$

Partial Derivative Practice

Example: *Find both partial derivatives for* $f(x,y) = (1 + x^3)y^2$.

$$\frac{\partial f}{\partial x} = (3x^2)(y^2)$$

[treat y's as constants]

$$\frac{\partial f}{\partial y} = (1+x^3)\cdot(2y)$$

[treat x's as constants]

Example: *Find both partial derivatives for* $f(x, y) = \underbrace{e^x}_{} \underset{\uparrow}{\underline{\sin(y)}}$.

$$\frac{\partial f}{\partial x} = e^x \sin(y)$$

$$\frac{\partial f}{\partial y} = e^x \cos(y)$$

Example: *Find both partial derivatives for* $f(x,y) = \frac{x^2}{4y}$. $= \frac{1}{4} x^2 \cdot \frac{1}{y}$

$= \left(\frac{1}{4} x^2\right) \cdot y^{-1}$

$$\frac{\partial f}{\partial x} = \frac{1}{4} \frac{1}{y} \cdot (2x)$$

$$\frac{\partial f}{\partial y} = \left(\frac{1}{4} x^2\right)\left(-y^{-2}\right)$$

$$= \frac{-x^2}{4y^2}$$

Partial Derivative Practice - 4

Question: $\frac{\partial}{\underline{\partial y}}\left(\overbrace{x^2\tan(\underline{y})+\underline{y}^2+x}^{f}\right)$ is

(a) $2x\sec^2(y)+2y+1$

[✓ (b) $x^2\sec^2(y)+2y$]

(c) $2x\tan(y)+1$

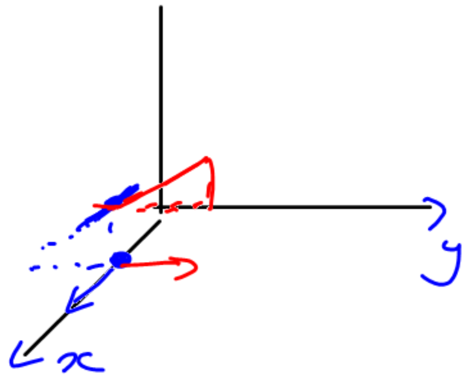
(d) $x^2\tan(y)+y^2+1$

$$\frac{\partial f}{\partial y} = x^2\cdot\sec^2(y)+2y+0$$

$$= x^2\sec^2(y)+2y$$

Example: *Consider* $f(x,y) = x^2 \sin y + e^{xy^2}$.

Find the value of $\dfrac{\partial f}{\partial x}(1,0)$. ← slope in x direction on surface @ $(1,0)$



$$\frac{\partial f}{\partial x} = 2x\cdot\sin(y) + e^{xy^2}\cdot(1\cdot y^2)$$

$$\frac{\partial f}{\partial x}(\underset{\underset{x}{\uparrow}}{1},\underset{\underset{y}{\uparrow}}{0}) = 2\cdot 1\cdot \overset{0}{\cancel{\sin(0)}} + e^{0}\cdot 0 = 0$$

Find the value of $\dfrac{\partial f}{\partial y}(1,0)$.

$$\frac{\partial f}{\partial y} = x^2\cdot\cos(y) + e^{xy^2}\cdot x\cdot 2y$$

$$\frac{\partial f}{\partial y}(1,0) = 1^2\cdot \overset{1}{\cancel{\cos(0)}} + e^{0}\cdot 1\cdot 2\cdot 0 = 1+0$$

$$= 1$$

What do these values partial derivative values tell you about the graph of $f(x,y) = x^2 \sin y + e^{xy^2}$ *near* $(1,0)$*?*

Slopes on surface in

direction of increasing x / positive x

or " y / positive y

Partial Derivatives - Economics

Example: *What are the signs of* $\frac{\partial g}{\partial x}(x,y)$ *and* $\frac{\partial g}{\partial y}(x,y)$ *if*

- $g(x,y)$ *is the amount of gas sold per week by a gas station,*
- x *is the station's price for gas, and*
- y *is the price charged by a nearby competitor?*

(a) $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are both positive.

(b) $\frac{\partial g}{\partial x}$ is positive, $\frac{\partial g}{\partial y}$ is negative.

[✓ (c) $\frac{\partial g}{\partial x}$ is negative, $\frac{\partial g}{\partial y}$ is positive.]

(d) $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are both negative.

our price incr

$\frac{\partial g}{\partial x} \to x$ incr, g $\underbrace{\text{decr}}$ / sales

negative

$\frac{\partial g}{\partial y} \to y$ incr, g $\underbrace{\text{incr}}$ / our sales

competitor

positive price incr

Example: *What would you expect the signs of* $\frac{\partial h}{\partial x}(x,y)$ *and* $\frac{\partial h}{\partial y}(x,y)$ *to be if*

- $h(x,y)$ *is the number of pairs of ski lift tickets sold in a year in Canada,*
- x *is the number of ski boots sold, and*
- y *is the number of tickets from Canada to warmer vacation spots.*

(a) $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are both positive.

[✓ (b) $\frac{\partial h}{\partial x}$ is positive, $\frac{\partial h}{\partial y}$ is negative.]

(c) $\frac{\partial h}{\partial x}$ is negative, $\frac{\partial h}{\partial y}$ is positive.

(d) $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are both negative.

Economics key words: *substitutes* and *complements*.

$\frac{\partial h}{\partial x} \to x$ incr , h incr

positive

$\frac{\partial h}{\partial y} \to y$ incr , h decr

negative

Partial Derivatives - Ideal Gas Law

Recall the ideal gas law, written with pressure as a function of temperature and volume:

$$P(V,T) = \frac{nRT}{V}$$

For one mole of gas ($n = 1$) and the SI units of kPa, liters and degrees Kelvin, find $\frac{\partial P}{\partial T}$ and $\frac{\partial P}{\partial V}$. Give the units of both derivatives.

$$P = \overset{1}{\cancel{n}} R \underbrace{T}_{} \cdot V^{-1}$$

$$\frac{\partial P}{\partial T} = R \cdot V^{-1}(1) = \frac{R}{V} \qquad \frac{kPa}{{}^\circ K}$$

$$\frac{\partial P}{\partial V} = RT\left(-V^{-2}\right) = \frac{-RT}{V^2} \qquad \frac{kPa}{\ell}$$

Evaluate both derivatives at $T = 300^o$ K and $V = 10$ liters. Use $R = 8.31$ L kPa /(K mol)

$$\frac{\partial P}{\partial T} = \frac{R}{V} = \frac{8.31}{10} = 0.831 \ \frac{kPa}{°K}$$

$$\frac{\partial P}{\partial V} = -\frac{RT}{V^2} = \frac{-(8.31)(300)}{(10)^2} \cong -24 \ \frac{kPa}{\ell}$$

Express the meaning of both $P_T(10\ L, 300^o\ K)$ *and* $P_V(10\ L, 300^o\ K)$ *in words.*

$$\frac{\partial P}{\partial T} = \underbrace{0.831}_{\text{positive}} \quad \text{kPa}/{}^\circ\text{K}$$

for 1 mol of gas

@ 300°K, w/ 10 l vol,

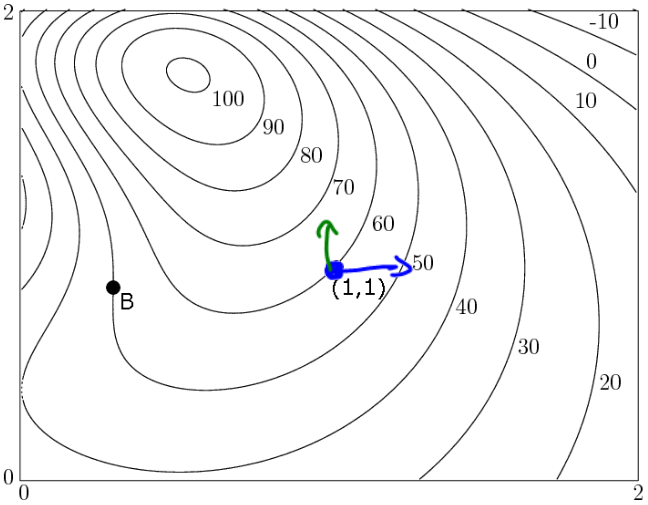
If we incr T, the pressure will increase by ≈ 0.831 kPa per °K of incr

(assuming volume remains constant)

$\frac{\partial P}{\partial V}$ indicates for an increase in volume, P will decrease by ≈ 24 kPa/l of volume increase.

Partial Derivatives from Contour Diagrams

Example: *Consider the contour diagram shown below, representing the function $h(x, y)$.*



as x incr

what happens to h(x) ?

$\frac{\partial h}{\partial y}$, as y incr

what happens to h?

→

pos/ incr x direction

Question: $\frac{\partial h}{\partial \boxed{x}}(1,1)$ is

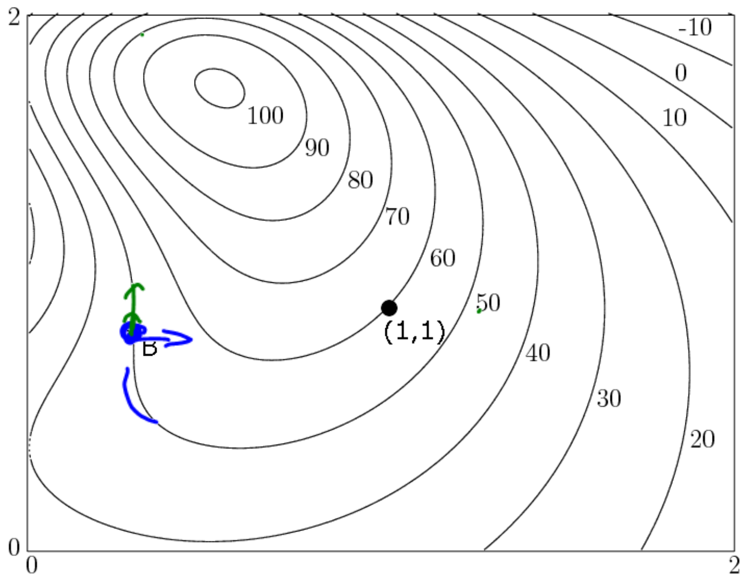
(a) positive [✓ (b) negative] (c) zero

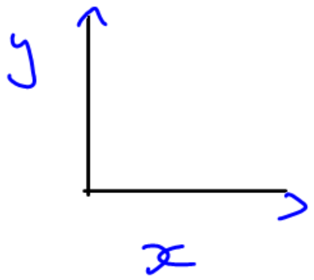
incr x → decr h ⇒ ∂h/∂x negative

Question: $\frac{\partial h}{\partial y}(1,1)$ is

[✓ (a) positive] (b) negative (c) zero

incr y ↑ → incr h ⇒ ∂h/∂y





Question: $\frac{\partial h}{\partial x}$ at the point B is

[✓ (a) positive] (b) negative (c) zero

incr x → incr in h

Question: $\frac{\partial h}{\partial y}$ at the point B is

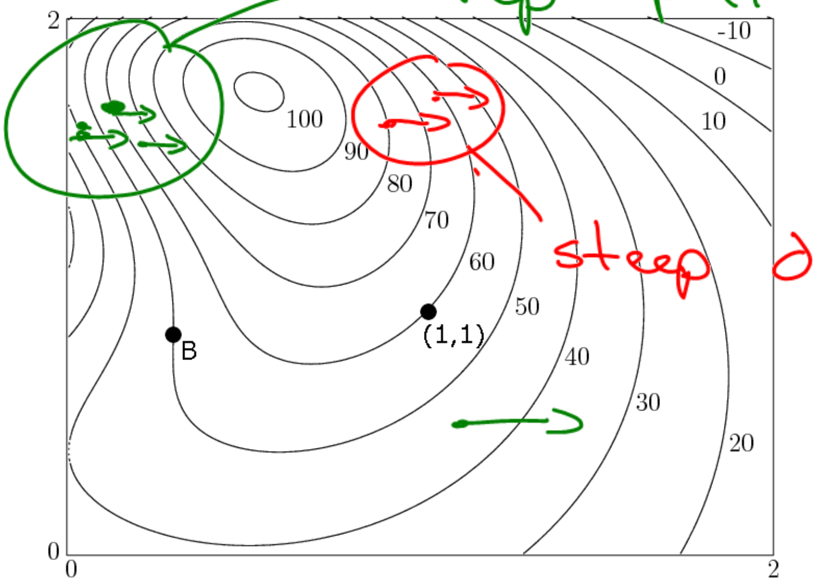
(a) positive (b) negative [✓ (c) zero]

small incr in y → small step → h is constant

($\Delta y \to 0$) stay on contour → $\Delta h = 0$

Steep uphill slopes in pos x direction

$\Rightarrow$ large $\frac{\partial h}{\partial x}$



steep downhill

→ large / negative $\frac{\partial h}{\partial x}$ values

On the same contour diagram, mark a point where $\dfrac{\partial h}{\partial x}$ *seems particularly large.*

large deriv → steep ^ slope
positive

→ incr in x → large change / incr in h.

→ tightly packed contours.

Partial Derivatives from Table Data

Given the following table of values for $f(x, y)$, calculate **approximate** *values for* $\dfrac{\partial f}{\partial x}(1,1)$ *and* $\dfrac{\partial f}{\partial y}(1,1)$. *(Multiple answers are possible, because we are estimating.)*

x

y	0.9	1.0	1.1	1.2
0.9	6.62	5.47	4.44	3.53
1.0	8.93	7.39	5.99	4.75
1.1	11.13	9.97	8.08	6.42
1.2	16.28	13.46	10.91	8.67

not a linear function

$$\frac{\partial F}{\partial x} \approx \frac{\Delta F}{\Delta x} = \frac{f_{end} - f_{start}}{x_{end} - x_{start}} = \frac{5.99 - 7.39}{1.1 - 1.0} = \frac{-1.4}{0.1} = -14$$

$$\frac{\partial f}{\partial y} \approx \frac{\Delta f}{\Delta y} = \frac{9.97 - 7.39}{1.1 - 1.0} = \frac{2.58}{0.1} = 25.8$$

What do the values $f_x(1,1)$ *and* $f_y(1,1)$ *tell you about the graph of* $f(x,y)$ *near* $(1,1)$*?*

$$\frac{\partial f}{\partial x} \approx -14$$

as we move incr x away from (1,1), f decr @ ≈ 14 units / unit incr in x

$$\frac{\partial f}{d y} \approx 25.8$$

as we move in incr y away from (1,1), f <u>incr</u> @ ≈ 25.8 units/unit of y