Week #6 : Laplace Transforms - Introduction

Goals:

- Introduction to the Laplace Transform
- Computing Laplace Transform with integration
- Using Laplace Transform Tables

Problem. Sketch a diagram showing the coverage of our solution techniques seen so far.

Integral transforms

An integral transform is an operator of the form

$$(Tf)(s) = \int_{t_1}^{t_2} K(s,t) f(t)\, dt \quad = F(s)$$

Transform of f(t)   ↑ has t and s in it

where the function $K(s,t)$ is called the ***kernel*** of the transform.

Problem. Give examples of other common integral transforms.

Laplace Transform

$$\int_0^\infty \underbrace{e^{-st}}_{K(s,t)} \cdot f(t)\, dt$$

Fourier Transform

$$\int_{-\infty}^{\infty} f(t) \underbrace{e^{-2\pi i s t}}_{K(s,t)}\, dt$$

Discrete Fourier Transform

z-transform

Problem. What direct purpose do Laplace transforms serve in a differential equations course?

Solve DEs w/ piecewise functions

w/ non-const coeffs.

Problem. What specific benefits are there to using Laplace transforms instead of our earlier methods?

enlarged pool of DEs that we can solve

Additional insight into patterns of solutions

Laplace Transform - Definition and Examples

Given a function $f(t)$ defined for $t \geq 0$, the ***Laplace transform*** of f is the function F defined by

finite

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)\, dt\,.$$

Transformed $F(s)$

Laplace transform

orig'l time-dependent $f(t)$

defn of improper integral

Problem. Based on our study of differential equations so far, what are the families of functions we have found in our solutions?

e^{at}, $\sin(bt)$, $\cos(bt)$

$t e^{at}$ $\qquad e^{at}\sin(bt)$ $\qquad e^{at}\cos(bt)$

1 $\qquad t$ $\qquad t^2$

We will investigate first how these functions will look after being transformed with the Laplace transform.

Problem. Find the Laplace transform of $f(t) = 1$.

$$F(s) = \underbrace{\mathcal{L}\{1\}}_{\text{def'n}} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

treat s as a constant

$$= \lim_{b\to\infty} \int_0^b e^{-st}\, dt = \lim_{b\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^b$$

$$= \lim_{b\to\infty} \frac{-1}{s}\left(\cancel{e^{-s\cdot b}}^{\to 0} - e^0\right)$$

$s > 0$

$$= -\frac{1}{s}(-1) = \frac{1}{s}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

t's s's

Problem. Find the Laplace transform of $f(t) = e^{at}$.

$$F(s) = \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

defn

intg'te

$$= \lim_{b\to\infty} \int_0^b e^{(a-s)t}\, dt = \lim_{b\to\infty} \left. \frac{e^{(a-s)t}}{(a-s)} \right|_0^b$$

$$= \lim_{b\to\infty} \frac{1}{a-s}\left(\cancel{e^{(a-s)b}}^{\to 0} - e^0\right) \qquad (a-s) < 0 \qquad s > a$$

$$= \frac{1}{a-s}(-1) = \frac{-1}{a-s} \frac{\times -1}{\times -1} = \frac{1}{-a+s} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{e^{-3t}\} = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

Problem. Evaluate $\mathcal{L}\{\sin(kt)\}$.

Hint. Recall that applying integration by parts twice will show

$$\underbrace{\int e^{-st}\sin(kt)}_{I}\,dt = \frac{-1}{s}e^{-st}\sin(kt) - \frac{k}{s^2}e^{-st}\cos(kt) - \frac{k^2}{s^2}\underbrace{\int e^{-st}\sin(kt)\,dt}_{I}$$

so $I\left(1+\frac{k^2}{s^2}\right) = \frac{-1}{s}e^{-st}\sin(kt) - \frac{k}{s^2}e^{-st}\cos(kt)$

$\downarrow \int_0^\infty$

$e^0=1$, $\cos(0)=1$, $\sin(0)=0$

$$\mathcal{L}(\sin(kt))\left(1+\frac{k^2}{s^2}\right) = \lim_{b\to\infty}\left.\left(-\frac{1}{s}e^{-st}\sin(kt) - \frac{k}{s^2}e^{-st}\cos(kt)\right)\right|_{t=0}^{t=\cancel{\infty}\,b}$$

$$= \lim_{b\to\infty}\left(-\frac{1}{s}e^{-sb}\sin(kb) - \frac{k}{s^2}e^{-sb}\cos(k\cdot b)\right) - \left(-\frac{k}{s^2}\right)$$

$\to 0$, $\to 0$, $s>0$

$$\mathcal{L}(\sin(kt))\left[1+\frac{k^2}{s^2}\right] = \frac{k}{s^2}$$

$$\mathcal{L}\{\sin(kt)\}\left[\frac{s^2+k^2}{s^2}\right] = \frac{k}{s^2}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{\cancel{s^2}} \cdot \frac{\cancel{s^2}}{s^2+k^2} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

eg. $\mathcal{L}\{\sin(3t)\} = \dfrac{3}{s^2+3^2} = \dfrac{3}{s^2+9}$

The transform of other common functions seen so far can also be found (though with more work, and almost always some integration by parts). The transforms are summarized in the table below.

Table of Laplace Transforms

$\int_0^\infty e^{-st} \cdot f(t)\, dt$

Function	Transform
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$

One important feature of the Laplace transform is that it is *linear*. This is a direct result of its definition as an integral, as integrals are also linear.

eg. $\mathcal{L}(5 \cdot 1) = \int_0^\infty e^{-st} \cdot 5 \, dt = 5\int_0^\infty e^{-st} \cdot 1 dt$

$= \frac{5}{s}$

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = aF(s) + bG(s)$$

const

Problem. Using the table, and the linearity of the transform, evaluate $\mathcal{L}\{4\cos(2t) - 6\sin(2t) - 5t^3\}$.

$k=2$ $k=2$ $n=3$

$$= 4\left(\frac{s}{s^2+2^2}\right) - 6\left(\frac{2}{s^2+2^2}\right) - 5\,\frac{3!}{s^4}$$

$$= \frac{4s}{s^2+4} - \frac{12}{s^2+4} - \frac{30}{s^4}$$

Function	Transform
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$

n factorial

Problem. Find the Laplace transform of $11 + 5e^{4t} - 6\sin(2t)$.

$$\mathcal{L}\{11 + 5e^{4t} - 6\sin(2t)\}$$

$$= 11 \cdot \frac{1}{s} + 5\,\frac{1}{s-4} - 6\left(\frac{2}{s^2+2^2}\right)$$

$$= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4}$$

Function	Transform
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$

The Inverse Laplace Transforms

As much fun as taking the forward Laplace transform is, the transformed versions aren't immediately useful (yet). We will also need to be able to *invert* the Laplace transform.

s's
$\downarrow$

Problem. Find a function whose Laplace transform is $\frac{2}{s^2}$, or $\underbrace{\mathcal{L}^{-1}}_{\text{inverse Laplace Transform}}\left\{\frac{2}{s^2}\right\}$

$\mathcal{L}^{-1}$: s $\to$ t's

$\mathcal{L}$: t $\to$ s's

$\frac{1}{s^2} \to \frac{1}{s^{1+1}}$

n=1

$\mathcal{L}(t^1) = \frac{1}{s^{1+1}} = \frac{1}{s^2}$

$\mathcal{L}(2t) = \frac{2}{s^2} = \frac{2}{s^2}$

$2t = \mathcal{L}^{-1}\left(\frac{2}{s^2}\right)$

Function	Transform
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$ $\leftarrow$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{5}{s-2}\right\}$

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$

$$\mathcal{L}^{-1}\left(\frac{5}{s-2}\right) = 5e^{2t}$$

Function	Transform
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$ ← $a=2$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$

Problem. Find $\mathcal{L}^{-1}\left\{\frac{1}{2s}+\frac{s}{s^2+4}\right\}$

Inverse Lapl' is also linear

$= \mathcal{L}^{-1}\left(\frac{1}{2}\,\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2+\underset{2^2}{4}}\right)$

$= \frac{1}{2}\cdot 1 + \cos(2t)$

Function	Transform
1	$\frac{1}{s}$ ←
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$ ← $k=2$
$\sin(kt)$	$\frac{k}{s^2+k^2}$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{2s-3}{s^2-s-6}\right\}$.

$$\frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{(s-3)}\frac{(s+2)}{(s+2)} + \frac{B}{(s+2)}\frac{(s-3)}{(s-3)}$$

$$2s-3 = A(s+2) + B(s-3)$$

$s=-2$: $\quad -4-3 = B(-5) \qquad B = \frac{5}{7}$

$s=3$: $\quad 6-3 = A(3+2) \qquad A = \frac{3}{5}$

$$\mathcal{L}^{-1}\left(\frac{2s-3}{s^2-s-6}\right) = \mathcal{L}^{-1}\left(\frac{3}{5}\,\frac{1}{s-3}\right) + \mathcal{L}^{-1}\left(\frac{5}{7}\,\frac{1}{s+2}\right)$$

$$= \frac{3}{5}e^{3t} + \frac{5}{7}e^{-2t}$$

Function	Transform
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$

Notes:

- A handout with a list of Laplace transforms is posted online. You **may** use this for tests, and it **will** be provided as part of the final exam.

- A handout reviewing partial fraction decomposition is posted online. (This will *not* be provided on the exam...)

Transforms of Exponential Products

In the functions we have worked with so far, there is a more complicated family that has not yet appeared: $\underbrace{f(t)e^{rt}}$.

Problem. Give examples of function in this family that we have seen in the class.

poly'ls

te^{rt}

e^{at}

cos/sin(kt)

t^2e^{rt}

e^{at} cos/sin(bt)

The rule from the table that applies to this family is

$$\mathcal{L}(f(t)e^{rt}) = F(s-a)$$

Reminder: you may use your table of Laplace transforms to answer this and future questions.

Problem. Find $\mathcal{L}(te^{3t})$.

Problem. Find $\mathcal{L}(\sin(2t)e^{-3t})$.

Problem. Find $\mathcal{L}(\cos(4t)e^{-3t})$.

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{2s+4}{s^2+2s+5}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+2s+5}\right\}$$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+4)^4}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+2s+5}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+2s+5}\right\} = 2\cos(2t)e^{-t} + \sin(2t)e^{-t}$$
$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+4)^4}\right\} = \frac{1}{2}t^2e^{-4t} - \frac{1}{3}t^3e^{-4t}$$

Problem. What do you notice about the the polynomials in the denominator and the form of the function we found in the inverse Laplace transform?

Problem. Compare the cases for r in a quadratic characteristic equation, and the cases for inverting a function with a quadratic denominator in s.

Inverse Laplace Transforms

Most inverse Laplace transforms can be evaluated using the following steps.

(1) Factor denominator completely (linear factors, and quadratic factors with complex roots)
(2) Use partial fractions to separate the factors into individual terms
(3) Invert each term, using the $F(s-a)$ form and completing the square as necessary.