Week #7 : Laplace - Step Functions, DE Solutions

Goals:

- Laplace Transform Theory
- Transforms of Piecewise Functions
- Solutions to Differential Equations
- Spring/Mass with a Piecewise Forcing function

Existence of Laplace Transforms

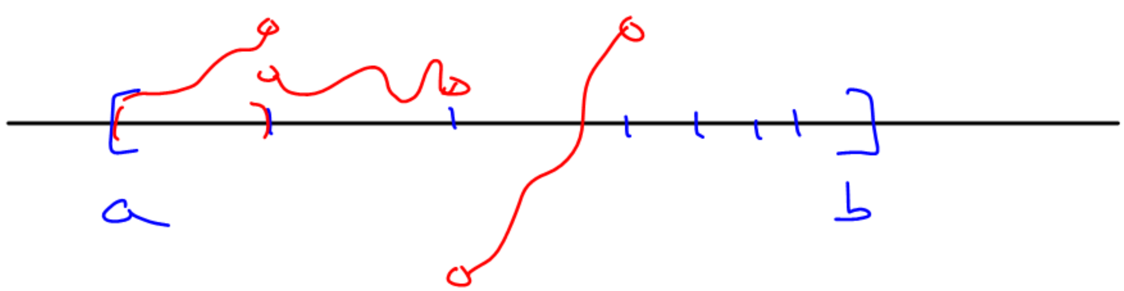
$F(s)$

Before continuing our use of Laplace transforms for solving DEs, it is worth digressing through a quick investigation of which functions actually *have* a Laplace transform.

A function f is ***piecewise continuous*** on an interval $t \in [a, b]$ if the interval can be partitioned by a finite number of points $a = t_0 < t_1 < \cdots < t_n = b$ such that

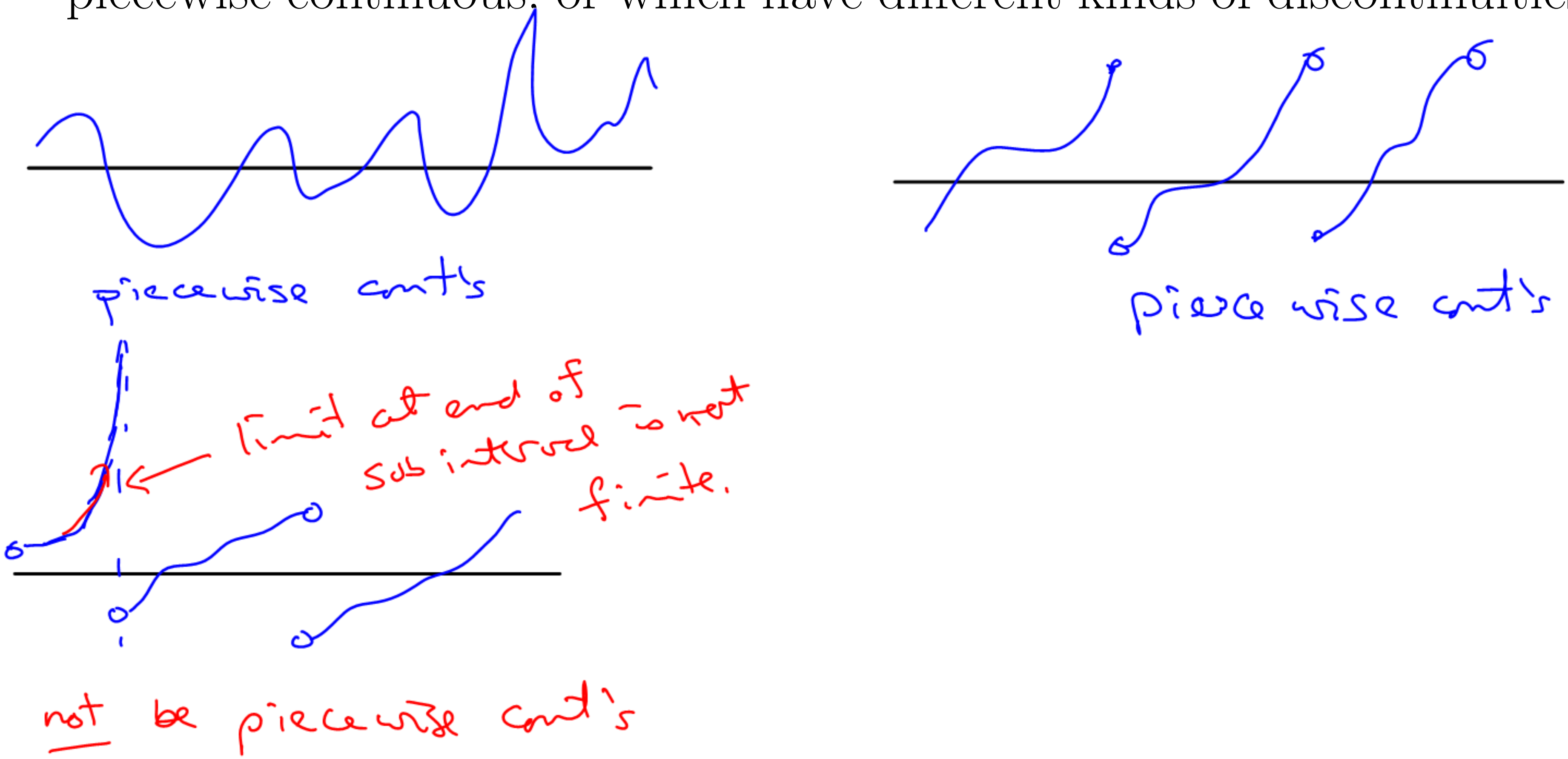
- f is continuous on each open subinterval (t_{i-1}, t_i).
- f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is continuous on $[a, b]$ except for a finite number of jump discontinuities. A function is piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.



$\int_0^\infty$

Problem. Draw examples of functions which are continuous and piecewise continuous, or which have different kinds of discontinuities.



One of the requirements for a function having a Laplace transform is that it be piecewise continuous. Classify the graphs above based on this criteria.

Another requirement of the Laplace transform is that the integral $\int_0^\infty e^{-st} f(t)\, dt$ *converges* for at least some values of s. To help determine this, we introduce a generally useful idea for comparing functions, "Big-O notation".

"OL"

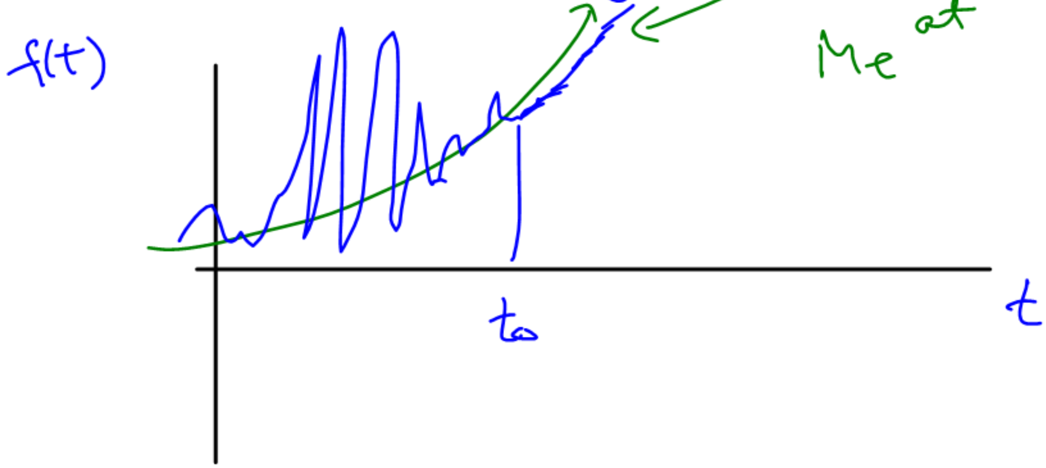
"on the order of"

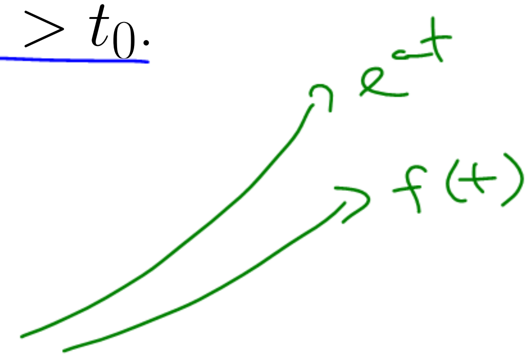
Big-O notation

We write $f(t) = O\left(e^{at}\right)$ as $t \to \infty$ and say f is ***of exponential order*** a (as $t \to \infty$) if there exists a positive real number M and a real number t_0 such that $|f(t)| \le Me^{at}$ for all $t > t_0$.

"f(t) is big O of e^{at}"

Me^{at}





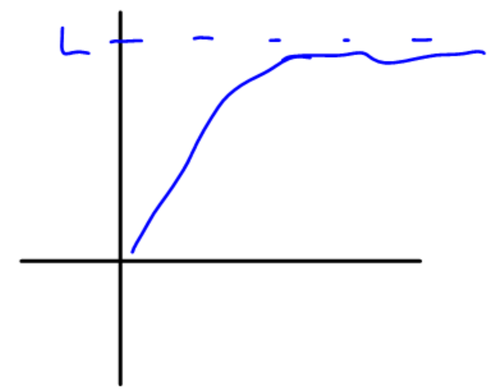
Lemma. *Assume* $\lim_{t\to\infty} \frac{|f(t)|}{e^{at}}$ *exists. Then*

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{at}} < \infty \qquad \text{finite}$$

if and only if $f(t) = O\big(e^{at}\big)$ *as* $t \to \infty$. $\square$

Problem. Show that bounded functions and polynomials are of exponential order a for all $a > 0$.

bounded f : $f \le L$ for all t



finite

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{at}} \le \lim_{t\to\infty} \frac{L}{\underbrace{e^{at}}_{\to\infty}} = 0$$

poly's: $\lim_{t\to\infty} \frac{\text{poly'l}}{e^{at}} = 0$

l'Hopital

both bounded and poly'l f's are of exp'l order.

Problem. Show that $e^{(t^2)}$ does **not** have exponential order.

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{at}} = \lim_{t\to\infty} e^{(t^2-at)} \to e^{\infty} \to \infty$$

e^{t^2} not of exp'l order

Problem. Are all the functions we have seen so far in our DE solutions of exponential order? Yes!

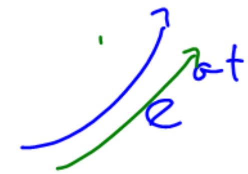
sin, cos : bounded ✓

poly's ✓

e^{at} ✓

The final reveal: what kinds of functions have Laplace transforms?

Proposition. *If f is*



- *piecewise continuous on $[0, \infty)$ and*
- *of exponential order a,*

then the Laplace transform $\mathcal{L}\{f(t)\}(s)$ exists for $s > a$.

The proof is based the comparison test for improper integrals.

Laplace Transform of Piecewise Functions

In our earlier DE solution techniques, we could not directly solve non-homogeneous DEs that involved piecewise functions. Laplace transforms will give us a method for handling piecewise functions.

$$y'' + ay' + by = \underbrace{\begin{matrix} \sin(at) & \cos(at) \\ e^{bt} & \text{poly} \end{matrix}}$$

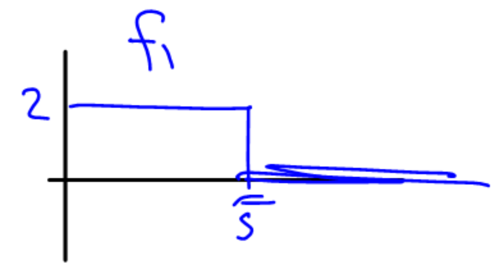
$$= \begin{cases} \sin(t) & 0 < t < 1 \\ \vdots & \end{cases}$$

Problem. Use the definition to determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 < t \le 5\,, \\ 0 & 5 < t \le 10\,, \\ e^{4t} & 10 < t\,. \end{cases}$$

$\to f_1 = 0$ for $t \geq 5$

$\to f_2$



$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_{t=0}^{t=\infty} e^{-st} f_1\, dt + \int_0^\infty e^{-st} \cdot f_2\, dt$$

$f_1 = 2$ on $t = 0..5$

$$= \int_0^{\boxed{5}} e^{-st} \cdot 2\, dt + \int_{10}^\infty \underbrace{e^{-st} \cdot e^{4t}}_{e^{-st+4t}}\, dt$$

shortcut improper

$$= 2 \cdot \frac{e^{-st}}{-s} \Bigg|_0^5 + \frac{e^{-st+4t}}{(-s+4)} \Bigg|_{10}^\infty$$

$$f(t) = \begin{cases} 2 & 0 < t \le 5\,, \\ 0 & 5 < t \le 10\,, \\ e^{4t} & 10 < t\,. \end{cases}$$

$$= \frac{2}{-s}\left(e^{-st}\right)\Big|_0^5 + \frac{1}{-s+4}\, e^{-st+4t}\Big|_{t=10}^{t=\infty}$$

$$= \frac{2}{-s}\left(e^{-5s} - 1\right) + \frac{1}{-s+4}\left(0 - e^{-s\cdot 10 + 4\cdot 10}\right)$$

$$= \frac{2}{s} - \frac{2e^{-5s}}{s} + \underbrace{\frac{1}{s-4}}_{\mathcal{L}(e^{4t})} e^{-10s}\cdot e^{40}$$

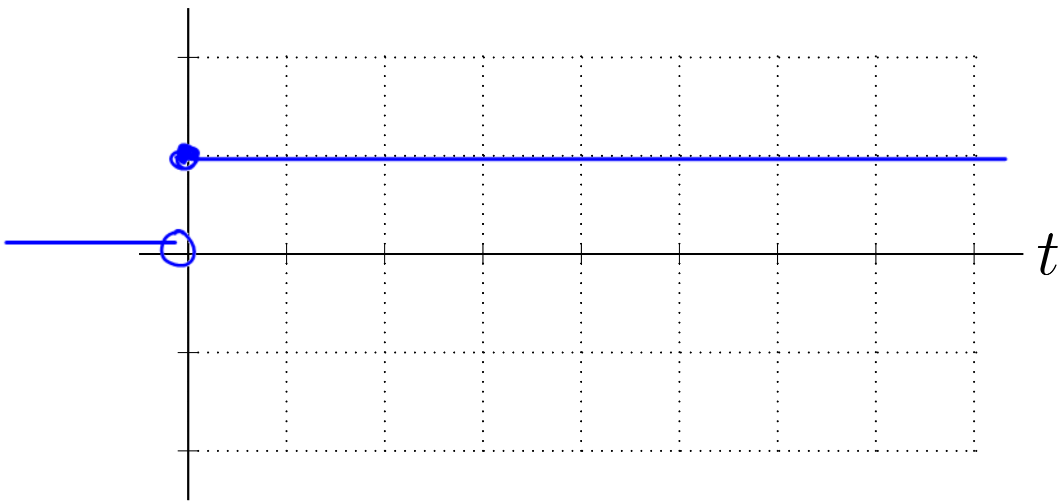
$\leftarrow$ $e^{\sim s}$ are indicators of $f(t)$ being piecewise.

Similar to previous

We would like avoid having to use the Laplace definition integral if there is an easier alternative. A new notation tool will help to simplify the transform process.

The ***Heaviside step function*** or ***unit step function*** is defined by $u(t) := \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$ off on

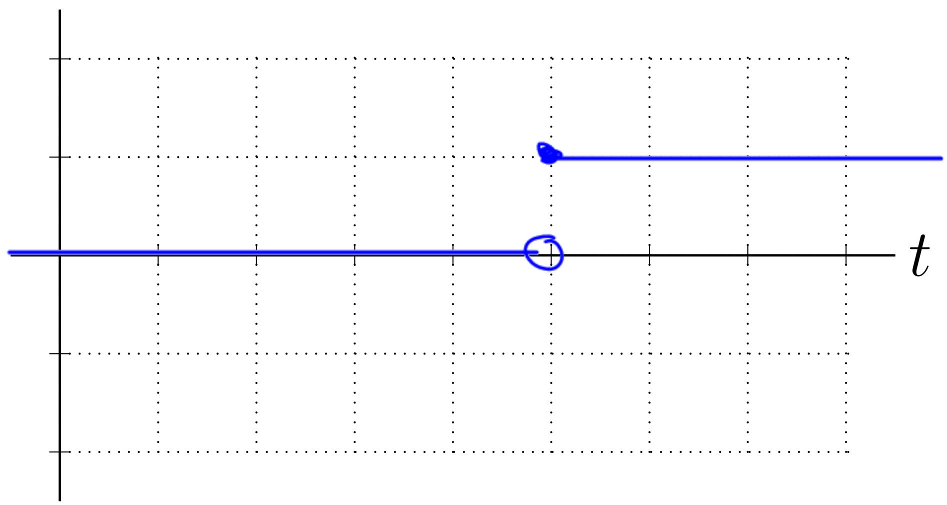
Problem. Sketch the graph of $u(t)$.



$$u(t) := \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

Problem. Sketch the graph of $u(t - 5)$.

shift graph right by 5



Laplace Transform Using Step Functions

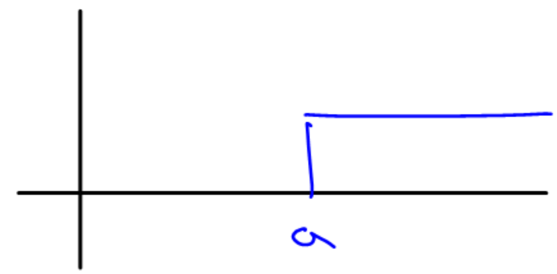
Problem. For $a > 0$, compute the Laplace transform of

$$u(t-a) = \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t \geq a. \end{cases}$$

$$\mathcal{L}(u(t-a)) \underset{\text{defn}}{=} \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$u = 0$ for $t < a$

$$= \int_{\underbrace{a}_{t \geq a}}^\infty e^{-st} \cdot 1 \, dt$$



$$= \frac{e^{-st}}{-s} \Bigg|_{a=t}^{\infty=t}$$

$\mathcal{L}\{1\}$

$\downarrow$

$$= \left(0 - \frac{e^{-s \cdot a}}{-s}\right) = \frac{1}{s} e^{-as}$$

step function

Laplace Transform of Step Functions

$$\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$$

An alternate (and more directly useful form) is

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

Notation: $u(t-a)$

$= u_a(t)$

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

t's replaced by t+a

Problem. Find $\mathcal{L}(u_2)$.

no t's to replace

$$\mathcal{L}\{u_2 \cdot 1\} = e^{-2s}\mathcal{L}\{1\}$$

$$= e^{-2s} \cdot \frac{1}{s}$$

Problem. Find $\mathcal{L}(u_\pi)$.

$$\mathcal{L}\{u_\pi\} = e^{-\pi s} \cdot \mathcal{L}\{1\}$$

$$= e^{-\pi s} \cdot \frac{1}{s}$$

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

Problem. Find $\mathcal{L}(tu_3)$.

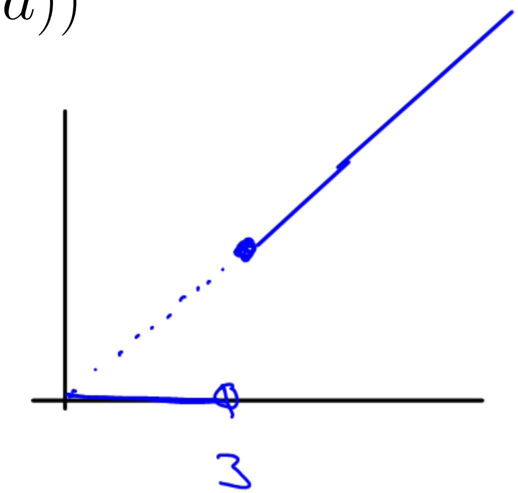
$\mathcal{L}\{3\} = \frac{3}{s}$

$\mathcal{L}(u_3 \cdot t) = e^{-3s}\mathcal{L}\{t+3\}$

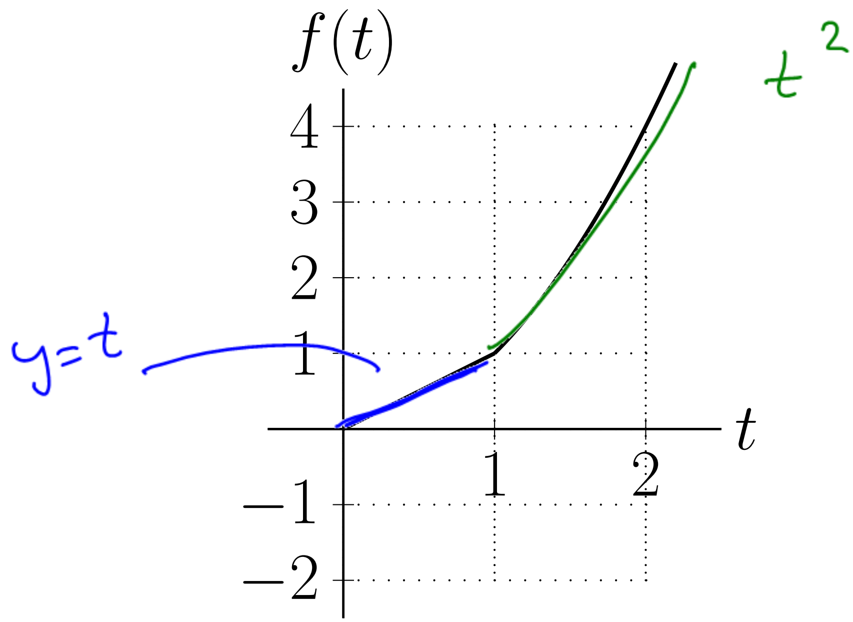
replace t's by t+3

$\mathcal{L}(t^1) = \frac{1!}{s^{1+1}}$

$= e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$



Problem. Here is a more complicated function made up of $f = t$ and $f = t^2$.



Write the function in piecewise form, and again using step functions.

t "on" at $t=0$

t "on" at $t=1$

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ t^2 & 1 < t \end{cases} = t(u_0 - u_1) + t^2 \cdot u_1$$

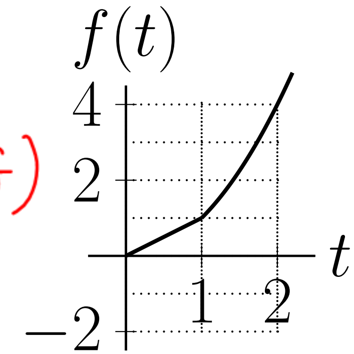
$t > 1$, $t \cdot (1-1)$

$= t \cdot 0$

$y = ?$

write w/ u's

Problem. Find $\mathcal{L}(t(u_0 - u_1) + t^2 u_1)$.



$$= \mathcal{L}(t \cdot u_0) - \mathcal{L}(t \cdot u_1) + \mathcal{L}(t^2 \cdot u_1)$$

$$\mathcal{L}(u_a \cdot f) = e^{-as} \cdot \mathcal{L}(f(t+a))$$

$$= \overset{1}{\cancel{e^{-0s}}} \cdot \mathcal{L}(t+0) - e^{-1s}\mathcal{L}(t+1) + e^{-1s}\mathcal{L}((t+1)^2)$$

expand

$$= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + e^{-s}\mathcal{L}(t^2 + 2t + 1)$$

$$= \frac{1}{s^2} \overbrace{- e^{-s}\left(\frac{1}{s^2} + \cancel{\frac{1}{s}}\right) + e^{-s}}^{\text{common}} \left(\frac{2}{s^3} + \cancel{2}\frac{1}{s^2} + \cancel{\frac{1}{s}}\right)$$

$$= \frac{1}{s} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right)$$

Inverse Laplace Transform of Step Functions

a first in table

$$\mathcal{L}^{-1}\left\{\underbrace{e^{-as}}F(s)\right\} = f(t-a)u_a$$

t's replaced by t−a

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$

$F(s) = \dfrac{1}{s^2}$

$= \mathcal{L}^{-1}\left\{e^{-2s} \cdot \dfrac{1}{s^2}\right\}$

$I(t)$

$= u_2(t)\,[t-2]$

step "on" at $t=2$

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s-4}\right\}$

$$= \mathcal{L}^{-1}\left\{ e^{-3s} \frac{1}{s-4} \right\}$$

$$\uparrow$$

$$\mathcal{L}(e^{4t})$$

$$= u_3(t) \cdot e^{4(t-3)}$$

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

Problem. Which of the following equals $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\}$?

1. $\frac{1}{s}\cos(\pi t)u_\pi$
2. $\frac{1}{\pi s}\cos(\pi(t-\pi))u_\pi$
3. $\frac{1}{2}\sin(2(t-\pi))u_\pi$
4. $\frac{1}{\pi}\sin(2(t-\pi))u_\pi$

$\mathcal{L}^{-1}\left\{e^{-\pi s} \cdot \frac{1}{2} \frac{\times 2}{s^2+4}\right\}$

$= \frac{1}{2} u_\pi(t) \sin(2(t-\pi))$

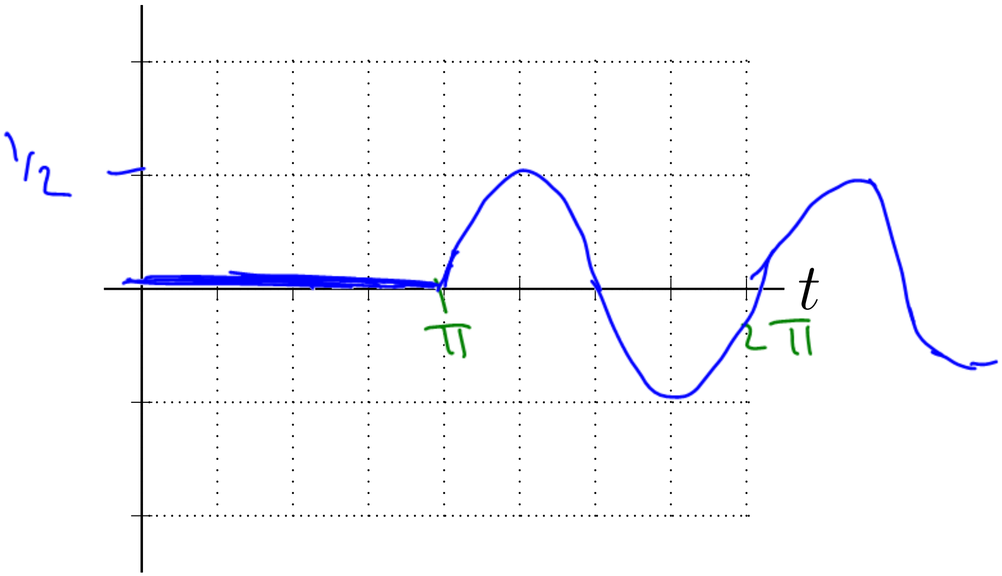
$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

Problem. Sketch the graph of

freq $\to$ period $= \frac{2\pi}{2} = \pi$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} = \frac{1}{2}\sin(2(t-\pi))u_\pi$$

"on" at $t = \pi$



$f(t)$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s-1)(s-2)}\right\}$

$\mathcal{L}^{-1}\{e^{-as}\cdot F(s)\}$
$= u_a \cdot f(t-a)$

$= \mathcal{L}^{-1}\left\{e^{-2s}\cdot\left[\dfrac{1}{(s-1)(s-2)}\right]\right\}$

↓ need part frac

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s-1)$$

$s=2$ $\quad 1 = B(2-1)$ $\quad \boxed{B=1}$

$s=1$ $\quad 1 = A(1-2)$ $\quad \boxed{A=-1}$

$\mathcal{L}^{-1}\left\{e^{-2s}\left[\left(\dfrac{-1}{s-1}\right)+\left(\dfrac{1}{s-2}\right)\right]\right\}$

$= -\mathcal{L}^{-1}\left\{e^{-2s}\dfrac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{e^{-2s}\dfrac{1}{s-2}\right\} = -u_2\, e^{1(t-2)} + u_2\, e^{2(t-2)}$

Tips for Inverse Laplace With Step/Piecewise Functions

- Separate/group all terms by their e^{-as} factor.

- Complete any partial fractions **leaving the e^{-as} out front** of the term.

  - The e^{-as} only affects final inverse step.

  - Partial fraction decomposition only works for polynomial numerators.

$$\frac{10\left(e^{-10s}\right)}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$= \left(\frac{A}{s-1} + \frac{B}{s-2}\right)\cdot e^{-10s}$$

The reason Laplace transforms can be helpful in solving differential equations is because there is a (relatively simple) transform rule for derivatives of functions.

Proposition (Differentiation)**.** *If f is continuous on $[0, \infty)$, $f'(t)$ is piecewise continuous on $[0, \infty)$, and both functions are of exponential order a, then for $s > a$, we have*

initial condition

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

s multiplier $\mathcal{L}$ of orig'l

Problem. Confirm the transform table entry for $\mathcal{L}\{\cos(kt)\}$ with the help of the transform derivative rule and the transform of $\sin(kt)$.

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \mathcal{L}\left\{\frac{d}{dt}\frac{1}{k}\sin(kt)\right\}$$

$$= \mathcal{L}\left\{\frac{1}{k}(\sin(kt))'\right\} = \frac{1}{k}\left[s\,\mathcal{L}\{\sin(kt)\} - \sin(0)\right]$$

$\mathcal{L}$ for deriv

$$= \frac{1}{\cancel{k}}\, s\left[\frac{\cancel{k}}{s^2+k^2} - 0\right]$$

$$= \frac{s}{s^2+k^2}$$

We can generalize this rule to the transform of higher derivatives of a function.

Theorem (General Differentiation). *If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous on $[0, \infty)$, $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, and all of these functions are of exponential order a, then for $s > a$, we have*

s power decr → f deriv incr

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f\}(s) - \underbrace{s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}_{\text{Series of initial cond's}}.$$

Can be proven using integration by parts n times.

n^{th} deriv

F(S)

s^n · Transform of orig'l f

Most commonly in this course, we will need specifically the transform of the second derivative of a function.

Corollary (Second Differentiation)**.** *If $f(t)$ and $f'(t)$ are continuous on $[0, \infty)$, $f''(t)$ is piecewise continuous on $[0, \infty)$, and all of these functions are of exponential order a, then for $s > a$, we have*

2 derivs s^2 ⟶ s power dec

$$\mathcal{L}\{f''(t)\}(s) = s^2 \mathcal{L}\{f\}(s) - sf(0) - f'(0)\,.$$

⟶

f deriv incr

Solving Initial Value Problems with Laplace Transforms

Problem. Sketch the general method.

$\mathcal{L}(y)$, unknown

DE $\quad y'' + ay' + by = f(t) \xrightarrow{\mathcal{L}} s^2 Y(s) + \ldots \quad Y(s) = \mathcal{L}(f(t))$

no derivs

$\downarrow$

y_c, y_{NH}

solve for $Y(s)$ $\downarrow$ algebraic eq'n

$\downarrow$

solution $\quad y = \ldots\ldots \xleftarrow{\mathcal{L}^{-1}} Y(s) = \ldots\ldots$

Problem. Find the Laplace transform of the entire DE

$$\mathcal{L}\left(x' + x\right) = \mathcal{L}\left(\cos(2t)\right), \; x(0) = 0$$

$\mathcal{L}(x(t)) = X(s)$

$$\underbrace{\left[sX(s) - x(0)\right]}_{\mathcal{L}(x')} + X(s) = \frac{s}{s^2+2^2}$$

$$sX(s) + X(s) = \frac{s}{s^2+4}$$

Problem. Note the form of the equation now: are there any derivatives left?

<u>No!</u>

Problem. Solve for $X(s)$.

$$sX(s) + X(s) = \frac{s}{s^2+4}$$

$$(s+1)\,X(s) = \frac{s}{s^2+4}$$

$$X(s) = \frac{s}{(s^2+4)(s+1)}$$

$$X(s) = \frac{s}{(s^2+4)(s+1)}$$

Problem. Put $X(s)$ in a form so that you can find its inverse transform. → partial fractions

$$\frac{s}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1}$$

$$s = (As+B)(s+1) + C(s^2+4)$$

$s=-1$ $\quad -1 = C(1+4) \qquad \boxed{C = -1/5}$

equate s^2 $\quad 0 = A + C \qquad A = -C \qquad \boxed{A = 1/5}$

const coff's $\quad 0 = B + 4C \qquad B = -4C \qquad \boxed{B = 4/5}$

$$X(s) = \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{1}{s^2+4} + \frac{-1}{5}\frac{1}{s+1}$$

Problem. Find $x(t)$ by taking the inverse transform.

$$X(s) = \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{1}{2}\frac{1 \times 2}{s^2+4} - \frac{1}{5}\frac{1}{s+1}$$

$\mathcal{L}^{-1}$

$$\frac{2}{s^2+2^2} \qquad \frac{1}{s-a}$$

$$\boxed{x(t) = \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) - \frac{1}{5}e^{-t}}$$

particular sol'n

b/c we used initial condition $x(0)=0$

Problem. Confirm that the function you found is a solution to the differential equation $x' + x = \cos(2t)$.

proposed sol'n

$$x(t) = \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) - \frac{1}{5}e^{-t}$$

$\downarrow$ need x'

$$x' = -\frac{2}{5}\sin(2t) + \frac{4}{5}\cos(2t) + \frac{1}{5}e^{-t}$$

Sub into LHS of eq'n:

$$\underbrace{\left(-\frac{2}{5}\cancel{\sin(2t)} + \frac{4}{5}\cos(2t) + \cancel{\frac{1}{5}e^{-t}}\right)}_{x'} + \underbrace{\left(\frac{1}{5}\cos(2t) + \cancel{\frac{2}{5}\sin(2t)} - \cancel{\frac{1}{5}e^{-t}}\right)}_{x}$$

$$= \cos(2t) = \text{RHS of DE} \checkmark$$

Problem. Solve $y'' + y = \sin(2t)$, $y(0) = 2$, and $y'(0) = 1$.

$\mathcal{L}$ of whole DE:

$\mathcal{L}(y(t)) = Y(s)$

$$\left[s^2 Y(s) - s(2) - 1\right] + Y(s) = \frac{2}{s^2+4}$$

$s^2 \mathcal{L}(y) - s\,y(0) - y'(0)$

Solve for $Y(s)$

$$(s^2+1)\,Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1)\,Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1}$$

$$y'' + y = \sin(2t),\ y(0) = 2, \text{ and } y'(0) = 1.$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{(s^2+1)}$$

$\downarrow$ part frac

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$s=0$ $\quad 2 = B + 4D$ ① $\qquad$ ① − ③ $\quad 2 = 3D$ $\quad \boxed{D = \frac{2}{3}}$

s^3: $\quad 0 = A + C$ ② $\qquad$ ③ $\rightarrow$ $\boxed{B = -D = \frac{-2}{3}}$

s^2: $\quad 0 = B + D$ ③

s: $\quad 0 = A + 4C$ ④ $\qquad$ ② − ④ $\quad 0 = -3C$ $\quad \boxed{C = 0}$

④ $\quad \boxed{A = 0}$

$$Y(s) = \overset{B}{\overset{\downarrow}{-\frac{2}{3}}}\,\frac{1}{2}\,\frac{\times 2}{s^2+4} + \overset{D}{\overset{\downarrow}{\frac{2}{3}}}\,\frac{1}{s^2+1} + \frac{2s+1}{s^2+1}$$

$$= -\frac{1}{3}\,\frac{2}{s^2+4} + 2\,\frac{s}{s^2+1} + \frac{5}{3}\,\frac{1}{s^2+1}$$

$\mathcal{L}^{-1} \downarrow$

$$y(t) = -\frac{1}{3}\sin(2t) + 2\cos(t) + \frac{5}{3}\sin(t)$$

particular solution.

$$y'' + y = \sin(2t),\ y(0) = 2, \text{ and } y'(0) = 1.$$

Problem. Confirm your solution is correct.

$$y = \overbrace{-\tfrac{1}{3}\sin(2t)}^{y_{NH}} + \underbrace{2\cos(t) + \tfrac{5}{3}\sin(t)}_{y_c}$$

Need y', y''

$$y' = -\tfrac{2}{3}\cos(2t) - 2\sin(t) + \tfrac{5}{3}\cos(t)$$

$$\Big\downarrow \tfrac{d}{dt}$$

$$y'' = +\tfrac{4}{3}\sin(2t) - 2\cos(t) - \tfrac{5}{3}\sin(t)$$

$$\text{LHS} = \underbrace{\left[\tfrac{4}{3}\sin(2t) - \cancel{2\cos(t)} - \cancel{\tfrac{5}{3}\sin(t)}\right]}_{y''} + \left[-\tfrac{1}{3}\sin(2t) + \cancel{2\cos(t)} + \cancel{\tfrac{5}{3}\sin(t)}\right]$$

$$y'' \qquad + \qquad y$$

$$= 1\sin(2t) = \text{RHS} \quad ✓$$

Problem. Solve $\underbrace{y'' - 2y' + 5y}_{2^{nd}\text{ order}} = -8e^{-t}$, $y(0) = 2$, and $y'(0) = 12$.

non-homog's

[could use y_{NH}, y_c]

$\mathcal{L}$ of DE

$$\underbrace{\left[s^2 Y(s) - s\,y(0) - y'(0)\right]}_{\mathcal{L}(y'')} - 2\underbrace{\left[s\,Y(s) - y(0)\right]}_{\mathcal{L}(y')} + 5\,Y(s) = -8\,\frac{1}{s+1}$$

Gather $Y(s)$ terms

$$Y(s)\left[s^2 - 2s + 5\right] - s\cdot 2 - 12 - 2(-2) = -8\,\frac{1}{s+1}$$

$$Y(s)\left[s^2 - 2s + 5\right] = \frac{-8}{s+1} + 2s + 8$$

$$Y(s) = \frac{-8}{(s+1)(s^2 - 2s + 5)} + \frac{2s+8}{(s^2 - 2s + 5)}$$

$$y'' - 2y' + 5y = -8e^{-t},\ y(0) = 2, \text{ and } y'(0) = 12.$$

$$Y(s) = \frac{-8}{(s+1)(s^2-2s+5)} + \frac{2s+8}{(s^2-2s+5)}$$

$$\frac{-8}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+5}$$

$$-8 = A(s^2-2s+5) + (Bs+C)(s+1)$$

$s=-1$ $\quad -8 = A(1+2+5)$ $\quad \boxed{A=-1}$

s^2 coeff $\quad 0 = A + B$ $\quad \boxed{B=1}$

const $\quad -8 = 5A + C$ $\quad \boxed{C = -8-5A = -3}$

$$Y(s) = \frac{-1}{s+1} + \frac{s}{s^2-2s+5} - 3\frac{1}{s^2-2s+5} + \frac{2s+8}{s^2-2s+5}$$

$$= \frac{-1}{s+1} + 3\frac{s}{s^2-2s+5} + 5\frac{1}{s^2-2s+5}$$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{s}{s^2-2s+5} + 5\,\frac{1}{s^2-2s+5}$$

Goal find $\mathcal{L}^{-1}(Y(s))$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{s}{(\underbrace{s^2-2s+1}_{(s-1)^2})-1+5} + 5\,\frac{1}{(s^2-2s+1)-1+5}$$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{(s-1)+1}{(s-1)^2+4} + 5\,\frac{1}{(s-1)^2+4}$$

(2)

② $\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right) = \cos(2t)$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{(s-1)}{(s-1)^2+2^2} + 8\,\frac{1}{2}\,\frac{\cancel{1}\;2}{(s-1)^2+2^2}$$

$\mathcal{L}^{-1}(F(s-a)) = e^{at}\mathcal{L}^{-1}(F(s))$

$\downarrow \mathcal{L}^{-1} \downarrow$

$$\boxed{y(t) = -e^{-t} + 3e^{t}\cos(2t) + 4e^{t}\sin(2t)}$$

$$y'' - 2y' + 5y = -8e^{-t},\ y(0) = 2, \text{ and } y'(0) = 12.$$

Problem. Confirm your solution is correct.

$$y(t) = -e^{-t} + 3e^{t}\cos(2t) + 4e^{t}\sin(2t)$$

d/dt

Need y', y'' for LHS:

$$y' = e^{-t} + 3e^{t}\cos(2t) - 6e^{t}\sin(2t) + 4e^{t}\sin(2t) + 8e^{t}\cos(2t)$$

$$= e^{-t} + 11e^{t}\cos(2t) - 2e^{t}\sin(2t)$$

d/dt

$$y'' = -e^{-t} + \underbrace{11e^{t}\cos(2t) - \underbrace{22e^{t}\sin(2t) - 2e^{t}\sin(2t)}_{-24} - 4e^{t}\cos(2t)}_{=7}$$

LHS: $\left(-e^{-t} + 7e^{t}\cos(2t) - 24e^{t}\sin(2t)\right)$

$-2\left(e^{-t} + 11e^{t}\cos(2t) - 2e^{t}\sin(2t)\right)$

$+5\left(-e^{-t} + 3e^{t}\cos(2t) + 4e^{t}\sin(2t)\right)$

$= -8e^{-t}$ = RHS

e^{-t}'s: $-1 - 2 - 5 = -8e^{-t}$

$e^{t}\cos(2t)$: $7 - 22 + 15 \stackrel{?}{=} 0$

$e^{t}\sin(2t)$: $-24 + 4 + 20 \stackrel{?}{=} 0$

Problem. Describe the scenario for a spring/mass system defined by the differential equation

$$my'' + cy' + ky = \begin{cases} 10t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

Can this system be solved in a straightforward way using our earlier solution techniques?

Predict the motion for the spring/mass using the values given, if the mass starts at equilibrium at $t = 0$.

$$y'' + 4y' + 20y = \begin{cases} 10t & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

$$y'' + 4y' + 20y = \begin{cases} 10t & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases}$$

$$y'' + 4y' + 20y = \begin{cases} 10t & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

Animation

Problem. Compare and contrast between our methods so far for solving higher-order differential equations.