Week #11 : Complex Eigenvalues, Applications of Systems

Goals:

- Solutions for the Complex Eigenvalue Case
- Further Applications of Systems of DEs

Complex Eigenvalue Case

First-order homogeneous systems have the standard form:

$$\boxed{\vec{x}\,' = A\vec{x}} \longrightarrow \text{eigenvalues} \to \text{charact'c poly'l}$$

What happens when the coefficient matrix A has non-real eigenvalues?

(Note: for the remainder of the course, we will use the more traditional "i" instead of $\sqrt{-1}$; it will simplify some of the notation.)

real imag.

Proposition. *If the real matrix A has complex conjugate eigenvalues $\alpha \pm \beta\, i$ with corresponding eigenvectors $\vec{a} \pm \vec{b}\, i$, then two linearly independent real vector solutions to $\vec{x}\,' = A\vec{x}$ are*

$r = 1,2$

α = real part

β = imaginary

$$\vec{x}_1 = e^{\alpha t}\Big[\underbrace{\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}}_{\text{new}}\Big] \quad \textit{and} \quad e^{\alpha t}\Big[\underbrace{\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}}_{\text{new}}\Big] = \vec{x}_2$$

earlier: $e^{\alpha t}\cos(\beta t)$ $\qquad$ $e^{\alpha t}\sin(\beta t)$

Problem. Solve $\vec{x}'(t) = A\vec{x}(t)$ where $A := \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$.

assumption $\vec{x}(t) = e^{\lambda t}\vec{v}$

Find eigenvalues of A: set $\det(A - rI) = 0$

$$\begin{vmatrix} -1-r & 2 \\ -1 & -3-r \end{vmatrix} = (-1-r)(-3-r) - (-1)(2)$$

$$= 3 + 4r + r^2 + 2$$

$$= \underbrace{r^2 + 4r + 5}_{\text{charac'c poly'l}} = 0$$

quadratic formula

$$r = \frac{-4 \pm \sqrt{4^2 - 4(5)}}{2}$$

$$= -2 \pm \frac{1}{2}\sqrt{-4}$$

complex roots

$$= -2 \pm \sqrt{-1}$$

$$\boxed{r = -2 \pm 1\,i}$$

α β

Aside:

$e^{(-2+i)t}$ is a sol'n

but not all real sol'n

$$\vec{x}_1(t) = e^{\alpha t}\left[\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right]$$
$$\vec{x}_2(t) = e^{\alpha t}\left[\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right]$$
$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \vec{x}(t)$$

$r = -2 \pm 1i$ (two eigenvalues)

find eigenvector for

$r = \alpha \underset{\uparrow}{+} \beta i$

(vector for $r = \alpha - \beta i$ will be the complex conjugate of this)

Solve $(A - rI)\vec{v} = \vec{0}$

Treat 'i' as constant, except $i^2 = -1$

$$\begin{bmatrix} -1-(-2+i) & 2 \\ -1 & -3-(-2+i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

$$\boxed{\vec{v} = \begin{bmatrix} 1 \\ -1/2 + 1/2\, i \end{bmatrix}}$$

$$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow$$

$(1-i)v_1 + 2v_2 = 0$

let $v_1 = 1$

$2v_2 = -1 + i$

$\boxed{v_2 = \frac{1}{2}(-1+i)}$

not obvious, but 3 true that one row is redundant.

$$\vec{x}_1(t) = e^{\alpha t}\left[\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right]$$

$$\vec{x}_2(t) = e^{\alpha t}\left[\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right]$$

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \vec{x}(t)$$

$$\vec{v} = \begin{bmatrix} 1 \\ -\tfrac{1}{2} + \tfrac{1}{2} i \end{bmatrix} \times 2 = \begin{bmatrix} 2 \\ -1 + 1i \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{a}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} i \quad \text{for } r = -2 + i$$

$\alpha = -2$ $\quad \beta = +1$

From matrix theory,

$$\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} i \text{ is eig vect. for } r_2 = -2 - i$$

Use sol'n formula to build the general solution

same freq

$$\vec{x}(t) = c_1 e^{-2t}\left[\cos(t)\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$+ c_2 e^{-2t}\left[\sin(t)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

DE: $$\underbrace{\vec{x}'(t)}_{LHS} = \underbrace{\begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \vec{x}(t)}_{RHS}$$

Problem. Verify your solution is correct.

$$x_1 = e^{-2t}\left[\cos(t)\begin{bmatrix}2\\-1\end{bmatrix} - \sin(t)\begin{bmatrix}0\\1\end{bmatrix}\right]$$

$$LHS = -2e^{-2t}\left[\cos(t)\begin{bmatrix}2\\-1\end{bmatrix} - \sin(t)\begin{bmatrix}0\\1\end{bmatrix}\right] + e^{-2t}\left[-\sin(t)\begin{bmatrix}2\\-1\end{bmatrix} - \cos(t)\begin{bmatrix}0\\1\end{bmatrix}\right]$$

$$= e^{-2t}\left[\cos(t)\underbrace{\begin{bmatrix}-4+0\\2-1\end{bmatrix}}_{\begin{bmatrix}-4\\1\end{bmatrix}} + \sin(t)\underbrace{\begin{bmatrix}0-2\\2+1\end{bmatrix}}_{\begin{bmatrix}-2\\3\end{bmatrix}}\right]$$

LHS = RHS

$$RHS = \begin{bmatrix}-1 & 2\\-1 & -3\end{bmatrix}\left(e^{-2t}\right)\left[\cos(t)\begin{bmatrix}2\\-1\end{bmatrix} - \sin(t)\begin{bmatrix}0\\1\end{bmatrix}\right]$$

$$= \left(e^{-2t}\right)\left[\cos(t)\begin{bmatrix}-4\\1\end{bmatrix} + \sin(t)\begin{bmatrix}-2\\+3\end{bmatrix}\right]$$

x_1 is a sol'n.

Problem. Solve $\vec{x}\,'(t) = A\vec{x}(t)$ where

$$A := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\vec{x}(t) = e^{rt}\vec{v}$$

$$\downarrow$$

$$r\vec{v} = A\vec{v}$$

Find eigenvalues $\longleftarrow$

$$\det(A - rI) = 0$$

$$\left|\begin{array}{ccc} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{array}\right| \begin{array}{cc} 1-r & 0 \\ 2 & -1-r \\ 3 & 2 \end{array}$$

charac'c poly'l

$$\overbrace{= (1-r)(1-r)(1-r) + 0 + 0 - \left[0 + (2)(-2)(1-r) + 0\right]}$$

$$= (1-2r+r^2)(1-r) + 4(1-r)$$

$$= -r^3 + 2r^2 - r + r^2 - 2r + 1 + 4 - 4r$$

$$= -r^3 + 3r^2 - 7r + 5 = 0$$

Guess: $r = 1$: $\quad -1 + 3 - 7 + 5 = 0$

so $(r-1)$ is a factor

$$(r-1)(-r^2 + 2r - 5) = 0$$

$$\downarrow \times -1$$

$$(r-1)(r^2 - 2r + 5) = 0$$

quad formula $\hookrightarrow$ $r = 1 \pm 2i$

complex pair

$$\vec{x}_1(t) = e^{\alpha t}\left[\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right]$$
$$\vec{x}_2(t) = e^{\alpha t}\left[\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right]$$
$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}\vec{x}(t)$$

Let $r_1 = 1$ (real distinct eigenvalue)

Find eigenvector

$(A - rI)\vec{v} = \vec{0}$

$$\begin{array}{c} \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}$$

3 $\quad 3v_1 + 2v_2 = 0$

$v_1 = \frac{-2}{3} v_2 \rightarrow \boxed{v_1 = -2}$

pick $\boxed{v_2 = 3}$

2 $\quad 2v_1 - 2v_3 = 0$

$v_3 = v_1 = -2$

so $\vec{v}_1 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \rightarrow$ one sol'n is

$$x_1(t) = e^{1 \cdot t} \cdot \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

$$\vec{x}_1(t) = e^{\alpha t}\left[\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right]$$

$$\vec{x}_2(t) = e^{\alpha t}\left[\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right]$$

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \vec{x}(t)$$

$r_{2,3} = 1 \pm 2i$

Find eigenvector for $r_2 = 1 \underset{3}{+} 2i$

$(A - rI)\vec{v} = \vec{0}$

$$\begin{bmatrix} 1-(1+2i) & 0 & 0 \\ 2 & 1-(1+2i) & -2 \\ 3 & 2 & 1-(1+2i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}$$

$$\begin{array}{c} 1 \\ 2 \\ \end{array} \begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 $\quad -2i\, v_1 = 0 \qquad \boxed{v_1 = 0}$

2 $\quad -2i\, v_2 - 2 v_3 = 0$

let $v_2 = 1$

$-2 v_3 = 2i$

$v_3 = -i$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{x}_1(t) = e^{\alpha t}\left[\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right]$$
$$\vec{x}_2(t) = e^{\alpha t}\left[\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right]$$
$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \vec{x}(t)$$

$$\vec{v}_2 = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{a}} + i \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{\vec{b}}, \quad r_2 = \underset{\alpha}{1} + \underset{\beta}{2} i$$

so $x_2(t) = e^{t}\left[\cos(2t)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right]$

and $\vec{x}_3 = e^{t}\left[\sin(2t)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \cos(2t)\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right]$

General sol'n = lin'r comb'n of x_1 (real r)
x_2, x_3 (complex r pair)

$$x(t) = c_1 e^{t}\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + c_2 x_2(t) + c_3 x_3(t)$$

$$\vec{x}_1(t) = e^{\alpha t}\left[\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}\right]$$

$$\vec{x}_2(t) = e^{\alpha t}\left[\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b}\right]$$

$$\vec{x}\,'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \vec{x}(t)$$

Non-Homogeneous Systems

We now expand our solution approach to include **non-homogeneous** first-order systems:

$$\overbrace{\boxed{\vec{x}\,'(t) = A\vec{x}(t)}}^{\text{homog's}} + \boxed{\vec{f}(t)}$$

To solve this non-homogeneous form, we will use the same method as we used for higher-order systems, called the

method of undetermined coefficients.

Problem. What are the steps in this method?

- finding diff'l family of $\vec{f}(t)$

  ↳ set of functions $\{f_1, f_2 \,..\, f_m\}$

  ↑ scalars

- build proposed solution

$$\vec{x}_{NH} = f_1(t)\,\vec{u}_1 + f_2(t)\,u_2 + f_3(t)\,u_3 + \,..\, + f_m(t)\,\vec{u}_m$$

are undetermined coeffs

Problem. Consider the DE system

non - homog's

$$\vec{x}' = \underbrace{\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}}_{A} \vec{x} + \begin{bmatrix} 10e^{2t} \\ 0 \end{bmatrix}$$

corresp'd homog's DE.

The eigenvalues and corresponding eigenvectors for the matrix A in the system above are

$$\lambda_1 = 5,\ \vec{v}_1 = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1,\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find the general solution to the *homogeneous* part of the system.

$$\vec{x}_c = c_1 e^{5t} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 10e^{2t} \\ 0 \end{bmatrix}$$

Diff'l family $\{e^{2t}\}$

Assume a form for the *particular* solution for the *non-homogeneous* form.

Assume $\vec{x}_{NH} = e^{2t} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\vec{u}}$

Sub in $\vec{x}_{NH}$, solve for u_1, u_2

$$2\cancel{e^{2t}}\vec{u} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \cancel{e^{2t}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \cancel{e^{2t}} \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 2u_1 \\ 2u_2 \end{pmatrix} = \begin{bmatrix} 6u_1 - 7u_2 \\ u_1 - 2u_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4u_1 + 7u_2 \\ -u_1 + 4u_2 \end{bmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

$$\begin{array}{c} ① \\ -4 \times ② \end{array} \begin{bmatrix} -4 & 7 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 0 & -9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$-9u_2 = 10$

$$\boxed{u_2 = \frac{-10}{9} \simeq -1.\overline{1}}$$

$$\vec{x}' = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 10e^{2t} \\ 0 \end{bmatrix}$$

Find the constants/coefficients in the particular solution.

$$u_2 = \frac{-10}{9}, \quad \begin{bmatrix} -1 & 4 \\ 0 & -9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$-u_1 + 4u_2 = 0$$

$$u_1 = 4u_2 = -\frac{40}{9}$$

so

$$\boxed{\vec{x}_{NH} = e^{2t} \begin{bmatrix} -\frac{40}{9} \\ -10/9 \end{bmatrix} = e^{2t} \cdot \frac{10}{9} \begin{bmatrix} -4 \\ -1 \end{bmatrix}}$$

$$\vec{x}(t) = \left[\ \vec{x}_c\ \right] + \left[\ \vec{x}_{NH}\ \right]$$

Write out the form for the general solution to

$$\vec{x}' = \underbrace{\begin{bmatrix}6 & -7\\ 1 & -2\end{bmatrix}}_{A}\vec{x} + \underbrace{\begin{bmatrix}2\\ -5t\end{bmatrix}}_{\text{experiment w/ } \vec{f}(t), \text{ non-homog's part}} \to \text{diff'l family } \{t, 1\}$$

in $\vec{x} = \vec{x}_c + \vec{x}_{NH}$ form.

$$\vec{x}(t) = \underbrace{c_1 e^{5t}\begin{bmatrix}7\\ 1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}}_{\vec{x}_c} + \underbrace{t\,\vec{u}_1 + 1\cdot\vec{u}_2}_{\vec{x}_{NH}}$$

consts

function / scalar

Write out the form for the general solution to

$$\vec{x}' = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} \sin(t) \\ \sin(2t) \end{bmatrix}$$

in $\vec{x} = \vec{x}_c + \vec{x}_{NH}$ form.

diff'Q family

$\{ \sin(t), \cos(t), \sin(2t), \cos(2t) \}$

$$\vec{x}(t) = \underbrace{c_1 e^{5t} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}_c} + \underbrace{\sin(t)\vec{u}_1 + \cos(t)\vec{u}_2 + \sin(2t)\vec{u}_3 + \cos(2t)\vec{u}_4}_{\vec{x}_{NH}}$$

Write out the form for the general solution to

$$\vec{x}' = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 10e^{-t} \end{bmatrix} \to \text{diff'l family } \{e^{-t}\}$$

in $\vec{x} = \vec{x}_c + \vec{x}_{NH}$ form.

same!

$$\vec{x} = \underbrace{c_1 e^{5t}\begin{bmatrix} 7 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}_c} + e^{-t}\vec{u}_1$$

Overlap b/w $\vec{x}_c$ and $\vec{x}_{NH}$ sol'ns.

$\Rightarrow$ requires special handling.

Repeated Solutions in $\vec{x}_c$ and $\vec{x}_{NH}$

If a member of the differential family needed for $\vec{x}_{NH}$, say $f(t)$, is already present in $\vec{x}_c$, then you must include

$$t \cdot f(t)$$

and all lower multiples of t in the assumed form for $\vec{x}_{NH}$.

new for systems

Problem. Write the form of the general solution to

non homog's

$$\vec{x}' = \underbrace{\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}}_{A} \vec{x} + \begin{bmatrix} 0 \\ 10e^{-t} \end{bmatrix}$$

diff'l family

$\{e^{-t}\}$

in $\vec{x}_c$ already

in $\vec{x} = \vec{x}_c + \vec{x}_{NH}$ form. Recall: for this A matrix,

$$\lambda_1 = 5,\ \vec{v}_1 = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1,\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x} = \underbrace{c_1 e^{5t}\begin{bmatrix} 7 \\ 1 \end{bmatrix} + c_2 e^{-1\cdot t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}_c} + \overbrace{t e^{-t}\vec{u}_1 + e^{-t}\vec{u}_2}^{\vec{x}_{NH}}$$

new func b/c of overlap w/ $\vec{x}_c$

vect const's

Find $\vec{u}_1$, $\vec{u}_2$ by subbing in $\vec{x}_{NH}$ into DE

Problem. Write out the form for the general solution to

$$\vec{x}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}}_{A} \vec{x} + \begin{bmatrix} \overset{F_{ext}}{\overset{\downarrow}{\sin(bt)}} \\ 0 \end{bmatrix}$$

diff'l family
$\{\sin(bt), \cos(bt)\}$

in $\vec{x} = \vec{x}_c + \vec{x}_{NH}$ form.

You are given that the eigenvalues of A are $\underbrace{\lambda_{1,2} = \overset{\alpha}{0} \pm \overset{\beta}{2}i}_{\text{eigenvalues}}$ and

$\vec{v}_{1,2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$ so

eigenvectors

$$\vec{x}_c = c_1 e^{\overset{\alpha}{0}t} \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(\overset{\beta}{2}t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) \right) + c_2 e^{0t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin(2t) \right)$$

pure oscillation w/ const amplitude

$\vec{x}_{NH} = \sin(bt)\vec{u}_1 + \cos(bt)\vec{u}_2$ $\underline{\underline{\text{if}}}$ $b \neq 2$

if $b=2$, $\sin(2t) / \cos(2t)$ in $\vec{x}_c \Rightarrow$

have to boost by 't':

$$\vec{x}_{NH} = \underline{\underline{t}} \cdot \sin(2t)\vec{u}_1 + \underline{\underline{t}}\cos(2t)\vec{u}_2 + \sin(2t)\vec{u}_3 + \cos(2t)\vec{u}_4$$

amplitudes grow linearly

Explain the conditions for resonance and beats in a first-order DE system.

non-homog's part has

- sin/cos w/ exactly the same freq as $\vec{x}_c$ → resonance

$\underset{\sim}{t}$ sin / $\underset{\sim}{t}$ cos in $\vec{x}_{NH}$

- sin/cos w/ close to same freq as in $\vec{x}_c$

↳ beats / large amplitude response

Modelling Systems - Interconnected Tanks

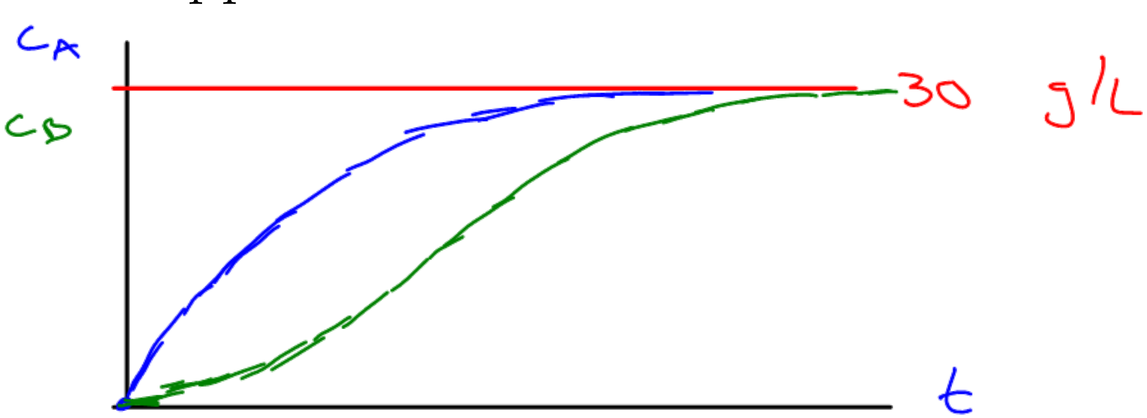
Consider the tanks shown below, which show water flowing between the tanks, and the concentration of a salt solution coming in. Within each tank, the water/salt solution is kept well mixed.

water

volumes of water are constant.

salt → dissolved

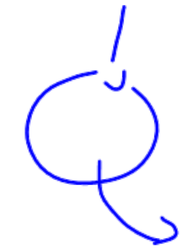
Problem. If both tanks start with no salt, sketch what you expect will happen to the concentration within each tank over time.



Problem. Create a system of differential equations that dictate how the two tank concentrations will evolve over time.

$$c = \frac{S}{V}$$

Study first $\quad s_A =$ mass of salt in tank A (g)



$$\left(\frac{g}{min}\right) \quad \frac{ds_A}{dt} = \left(\text{rate of salt in}\right) - \left(\text{rate salt going out}\right)$$

$$= \left(5 \; \frac{L}{min}\right)\Big(\overbrace{30 \; \frac{g}{L}}^{\text{conc}}\Big) - \left(5 \; \frac{L}{min}\right)\left(\frac{s_A}{V_A} \; \frac{g}{L}\right)$$

$$\frac{ds_A}{dt} = 150 \; \frac{g}{min} - 5 \; \frac{s_A}{V_A} \; \frac{g}{min}$$

To convert to

$$c_A = \frac{s_A}{V_A}$$

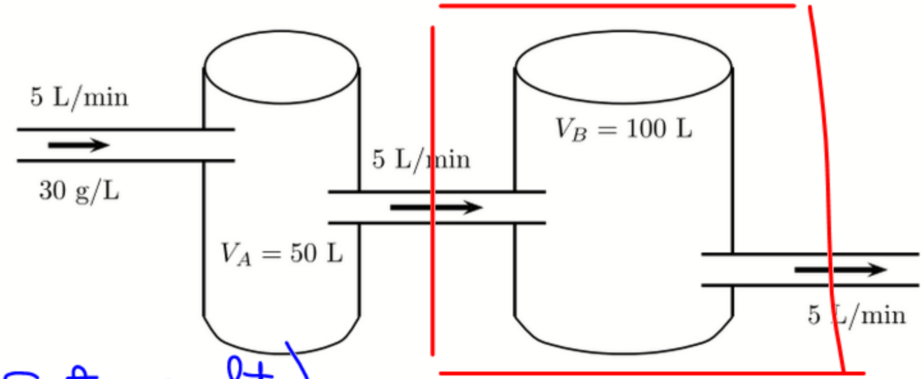
const $\rightarrow V_A$ $\quad d/dt$

$$\frac{dc_A}{dt} = \frac{1}{V_A} \frac{ds_A}{dt}$$

$$\boxed{\frac{dc_A}{dt} = \frac{1}{V_A}\left(150 - 5 \; \cancel{\frac{s_A}{V_A}}\, c_A\right)}$$

hong's

non-hong's



$$\frac{dc_B}{dt} = \frac{1}{V_B}\left(\underset{\text{in}}{\text{rate salt}}\ \text{g/min} - \underset{\text{out}}{\text{rate salt}}\ \text{g/min}\right)$$

$$\frac{dc_B}{dt} = \frac{1}{V_B}\left(5\,\frac{\text{L}}{\text{min}} \cdot c_A\,\frac{\text{g}}{\text{L}} - 5\,\frac{\text{L}}{\text{min}}\,c_B\right)\frac{\text{g}}{\text{L}}$$

$$\boxed{\frac{dc_B}{dt} = \frac{1}{V_B}\left(5c_A - 5c_B\right)}$$

matrix form

$$\underbrace{\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix}}_{\vec{x}'(t)} = \underbrace{\begin{bmatrix} -5/V_A & 0 \\ 5/V_B & -5/V_B \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} c_A \\ c_B \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 150/V_A \\ 0 \end{bmatrix}}_{\vec{f}(t)}$$

Problem. Predict the exact salt concentrations over time by solving the system of linear differential equations

$$\frac{d}{dt}\underbrace{\begin{bmatrix} c_A \\ c_B \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} \frac{-1}{10} & 0 \\ \frac{1}{20} & \frac{-1}{20} \end{bmatrix}}_{A} \begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Goal: $c_A = \ldots$

$c_B = \ldots$

find $\vec{x}_c$

A has eigenvalues $r_1 = -\frac{1}{10}$, $r_2 = -\frac{1}{20}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}_c = c_1 e^{-t/10} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t/20} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\vec{x}_{NH} = 1 \cdot \vec{u}$ sub into DE

const

$$\frac{d}{dt}\left(\cancel{\vec{u}}\right) = \begin{bmatrix} -\frac{1}{10} & 0 \\ \frac{1}{20} & -\frac{1}{20} \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$\vec{0}$

$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} \frac{-1}{10} & 0 \\ \frac{1}{20} & \frac{-1}{20} \end{bmatrix}\begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

diff'l family $\{1\}$

$$0 = -\frac{u_1}{10} + 3 \rightarrow \boxed{u_1 = 30}$$

$$0 = \frac{1}{20}u_1 - \frac{1}{20}u_2 \qquad \boxed{u_2 = u_1 = 30}$$

$$\vec{x}_{NH} = \begin{bmatrix} 30 \\ 30 \end{bmatrix} \text{g/L}$$

$\begin{bmatrix} c_A \\ c_B \end{bmatrix}$

w/ $t \to \infty$

$$\begin{bmatrix} c_A \\ c_B \end{bmatrix} = c_1 \underbrace{e^{-t/10}}_{\to 0}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \underbrace{e^{-t/20}}_{\to 0}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 30 \\ 30 \end{bmatrix} \rightarrow \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

$c_A(0)=0$

solve for c_1, c_2

$c_B(0)=0$

$$\begin{bmatrix} c_A \\ c_B \end{bmatrix} = c_1 e^{-t/10} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t/20} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

sub in $c_A=0$

$c_B=0$

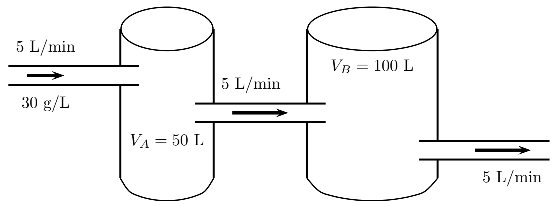
$t=0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

$0 = c_1 + 30$ $\qquad \boxed{c_1 = -30}$

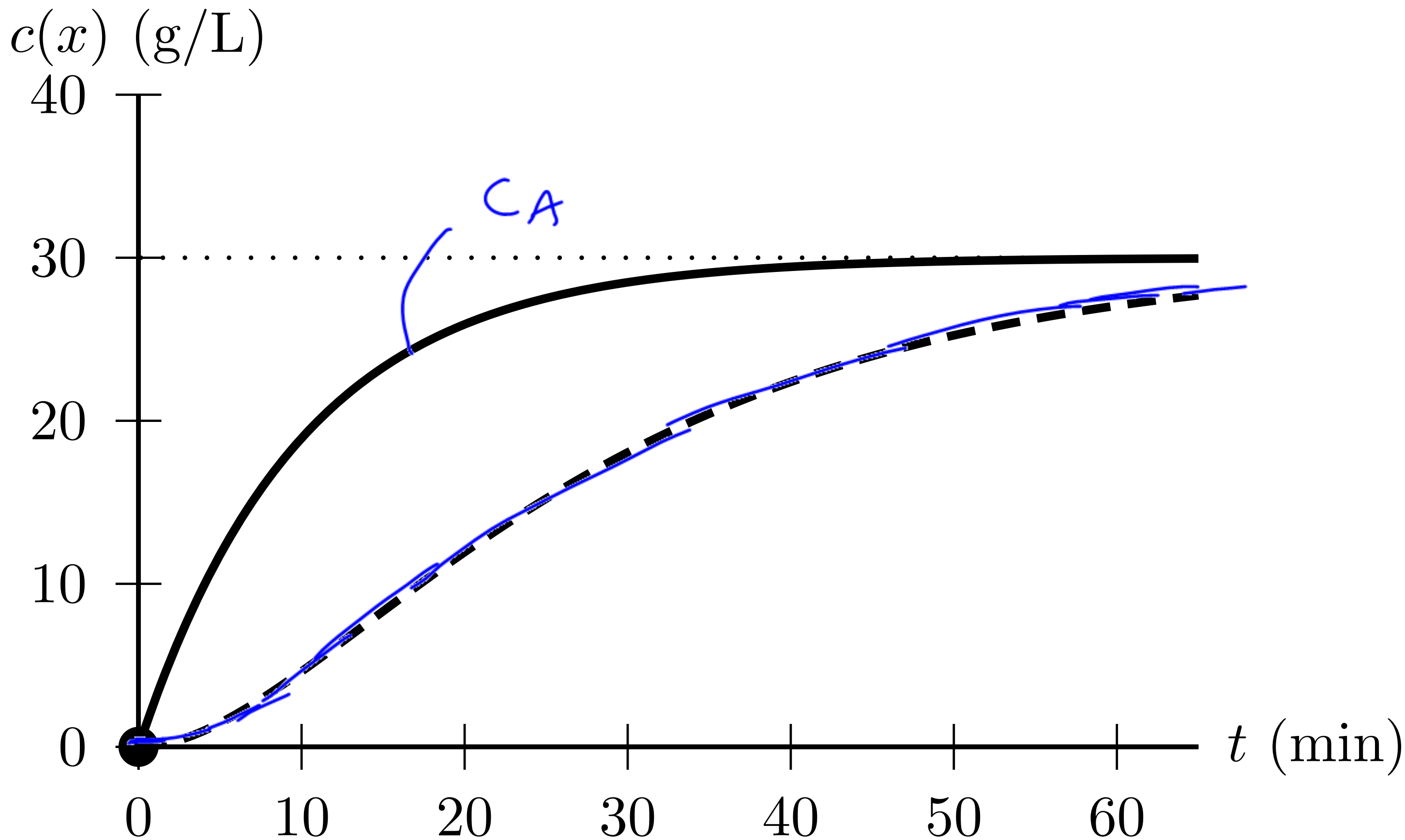
$0 = -c_1 + c_2 + 30$ $\qquad \boxed{c_2 = -30 + c_1 = -60}$

$$\begin{bmatrix} c_A \\ c_B \end{bmatrix} = -30\, e^{-t/10} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 60\, e^{-t/20} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

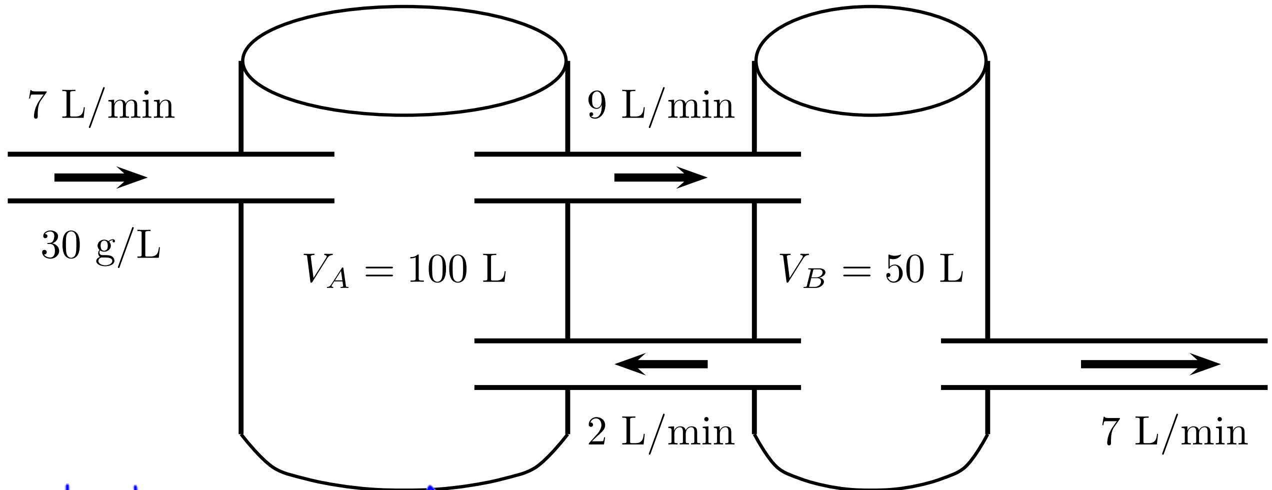


$$c_A = -30e^{-t/10} + 30;$$

$$c_B = 30e^{-t/10} - 60e^{-t/20} + 30$$



Consider the more complicated tank arrangement shown below.

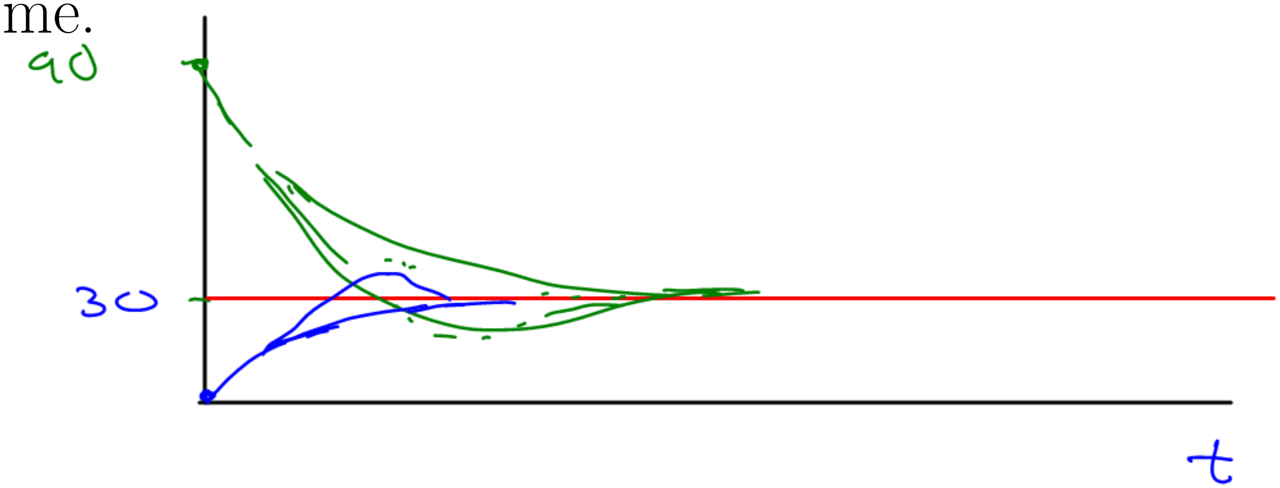


starts w/ no salt

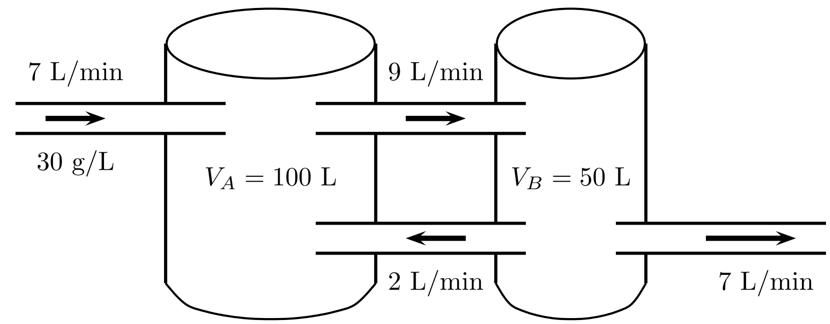
starts very concentrated

Problem. Given that the initial concentrations are $c_A(0) = 0$ g/L and $c_B(0) = 90$ g/L, ← initial conditions

sketch what you would predict for the concentration in each tank over time.



Problem. Construct the differential equation for the salt concentration in each tank, and write it in matrix form.



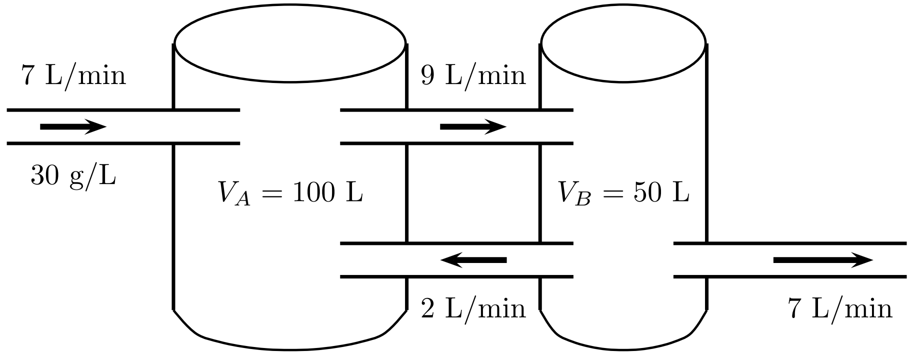
$$\frac{dc_A}{dt} = \frac{1}{V_A}\left(\underset{\text{g/min}}{\text{salt rate in}} - \underset{\text{g/min}}{\text{salt rate out}} \right)$$

$$\frac{dc_A}{dt} = \frac{1}{\cancel{V_A}\,100}\left(\left[\underset{\text{L/min} \times}{7} \times \underset{\text{g/L}}{30} + \underset{\frac{\text{L}}{\text{min}}}{2} \cdot \underset{\frac{\text{g}}{\text{L}}}{c_B} \right] - \left(\underset{\frac{\text{L}}{\text{min}}}{9} \cdot \underset{\text{g/L}}{c_A} \right) \right)$$

$$c_A' = -0.09\, c_A + 0.02\, c_B + 2.1$$

$$\frac{dc_B}{dt} = \frac{1}{V_{\cancel{B}}\,50}\left(9 \cdot c_A - \left(\overset{9}{\cancel{2+7}}\right) c_B \right)$$

$$c_B' = 0.18\, c_A - 0.18\, c_B$$



$$c_A' = -0.09\, c_A + 0.02\, c_B + 2.1$$

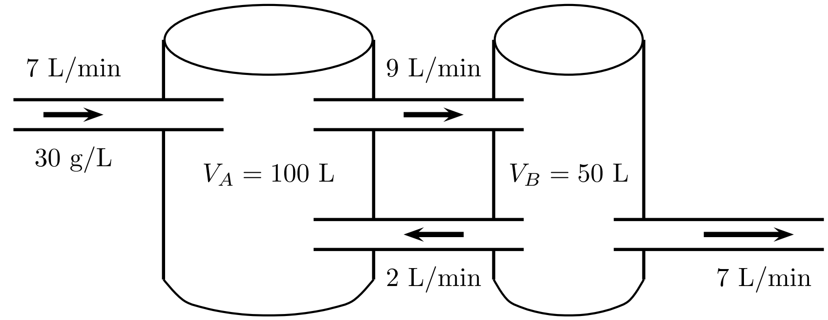
$$c_B' = 0.18\, c_A - 0.18\, c_B$$

$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix}\begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$$

Problem. Predict the exact salt concentrations over time by solving the system of linear differential equations

$$\frac{d}{dt}\underbrace{\begin{bmatrix}c_A\\c_B\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}-0.09 & 0.02\\0.18 & -0.18\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}c_A\\c_B\end{bmatrix}}_{\vec{x}} + \begin{bmatrix}2.1\\0\end{bmatrix}$$

↑ non-homog's

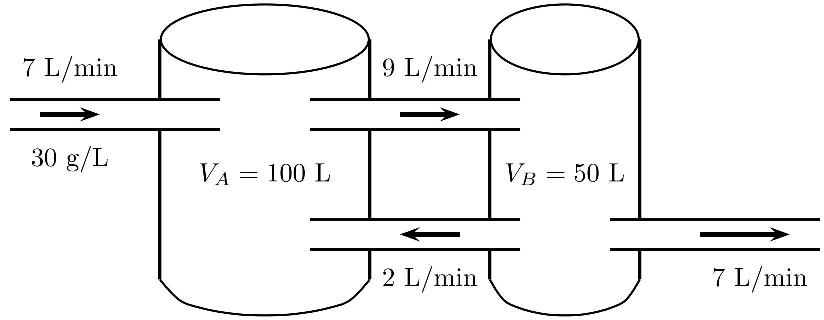


$\vec{x}_c$: eigenvalues $r_1 = -0.06$ $r_2 = -0.21$

eigenvectors $\vec{v}_1 = \begin{bmatrix}2\\3\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}1\\-6\end{bmatrix}$

$$\vec{x}_c = c_1 e^{-0.06t}\begin{bmatrix}2\\3\end{bmatrix} + c_2 e^{-0.21t}\begin{bmatrix}1\\-6\end{bmatrix}$$

$\to 0$ $\to 0$ as $t \to \infty$



$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix}\begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$$

diff'l family

$\{1\}$

Assume $\vec{x}_{NH} = \vec{u}$

Sub into DE

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cancel{\frac{d}{dt}(\vec{u})} = \begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$$

2x

+

$$\begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -2.1 \\ 0 \end{bmatrix}$$

$$-0.14\, u_2 = -4.2$$

$$\boxed{\begin{aligned} u_2 &= 30 \\ \ldots\quad u_1 &= 30 \end{aligned}}$$

g/L

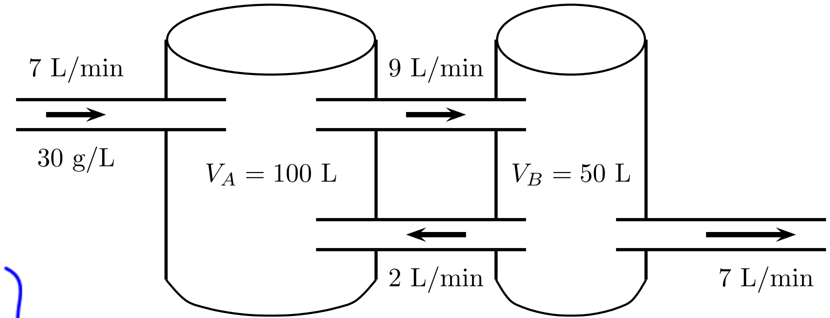
$$\vec{x}_{NH} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

$$x(t) = c_1 e^{-0.06t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-0.21t}\begin{bmatrix} 1 \\ -6 \end{bmatrix} + \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

Match c_1, c_2 to $t=0$ in sol'n

$c_A(0) = 0$

$c_B(0) = 90$



$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix}\begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$$

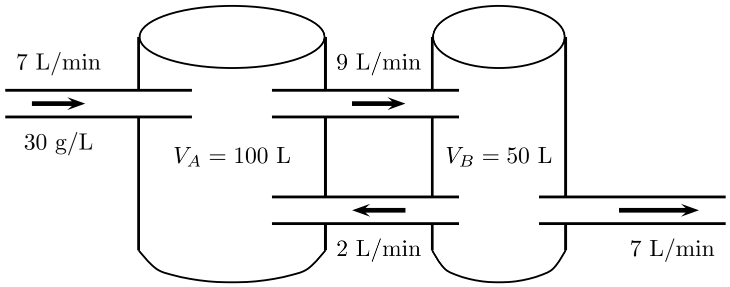
$$\begin{bmatrix} 0 \\ 90 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} c_1 + \begin{bmatrix} 1 \\ -6 \end{bmatrix} c_2 = \begin{bmatrix} -30 \\ 60 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -30 \\ 60 \end{bmatrix} \quad \ldots \quad c_1 = -8$$

$$c_2 = -14$$

Solution



eigenvectors

$$c_A = -8\boxed{(2)}e^{-0.06t} - 14\boxed{(1)}e^{-0.21t} + 30$$
$$c_B = -8\boxed{(3)}e^{-0.06t} - 14\boxed{(-6)}e^{-0.21t} + 30$$