Goals:

- Introduce implicit differentiation.
- Study problems involving related rates.
Implicit Differentiation

If we define a graph by the relationship $y = f(x)$, then we have a formula for tangent lines.

**Question:** What function below is the tangent line to $y = e^{-x}$ at $x = 0$?

(a) $y = -x + 1$

(b) $y = -(x - 1) + 1$

(c) $y = -x + e$

(d) $y = x + 1$

(e) $y = -(x - 1) - 1$
**Question:** Can we define a function in the form $y = f(x)$ that describes the points on the circle below?

(a) Yes.

(b) No.

(c) Maybe.
Write out a formula for the points in the circle shown.
Question: Can we, in principle, define a tangent line to the circle at the point $(2, \sqrt{12})$?

(a) Yes.

(b) No.

(c) Maybe.
Implicit Differentiation - Example 1

*Find a formula for the slopes of tangent lines to the circle $x^2 + y^2 = 16$.*

To find these slopes, we introduce a technique called *implicit differentiation*. The name comes from treating $y$’s in our formulas as implicit functions of $x$ when we differentiate, even though we don’t explicitly have $y$ written as a function.
For the circle $x^2 + y^2 = 16$, find the slope at the point $(2, \sqrt{12})$, and sketch it on the graph.
What is different about the derivative found by implicit differentiation, compared to the derivative of a function?
Implicit Differentiation - Example 2

Example: Sketch the curve defined by the relationship $x = y^2$.

Find the equation of the tangent line to this graph at the point $(4, 2)$. 
Tangent at (4,2) to $x = y^2$ (continued)
Implicit Differentiation - Example 3

Note that implicit derivatives of even complicated relationships are straightforward to compute.

**Example:** Use implicit differentiation to calculate \( \frac{dy}{dx} \) when

\[
x^3 + 2x^2y + \sin(xy) = 1.
\]
\[ x^3 + 2x^2y + \sin(xy) = 1 \]

*Is the point (1, 0) on this curve?*
\[ x^3 + 2x^2y + \sin(xy) = 1 \]

*Sketch the curve locally around the point \((1, 0)\).*
\[ x^3 + 2x^2y + \sin(xy) = 1 \]

*Estimate the $y$ location of the nearby point on the graph at $x = 0.95$.***
Here is the graph of the relation, shown at two different zoom levels.

Sketch the tangent line found in the previous question.
Related Rates

We can use the Chain Rule and Implicit Differentiation to solve problems involving related rates. As the name suggests, we use the rate of change (i.e., the derivative) of one function to calculate the rate of change (derivative) of a second function.
Example: A car starts driving north at a point 150 meters east of an observer at point A. The car is traveling at a constant speed of 20 meters per second. How fast is the distance between the observer and car changing after 10 seconds?
(Continued)
By using implicit differentiation, we can solve related rates problems even if we do not have an explicit formula for the function in terms of the independent variable (usually time).

**Example:** A conical water tank with a top radius of 2 meters and height 4 meters is leaking water at 0.5 liters per second. How fast is the height of the water in the tank changing when the remaining water in the tank is at a height of 2 meters?
Question: When the water height is above $2$ m, is the water height changing more quickly or more slowly than the value we just found?

(a) more quickly

(b) more slowly
General method for Related Rates problems

- Draw a picture (if possible) of the situation, and label all relevant variables.
- Identify which of the variables and their rate of change are known.
- Identify which rate you are trying to determine.
- Write an equation involving the changing variables, including the function whose rate you are trying to find.
- Apply implicit differentiation to the equation, *before* substituting any known variables.
- Substitute known variables and rates.
- Solve for desired rate.
No calculus course is complete without a related rates ladder problem.

**Example:** A 5 m ladder is propped against the wall. You climb to top, but then it starts to slip; the tip of the ladder is moving downwards at 1 m/s. Find the rate at which the bottom end of the ladder is sliding along the ground, when the bottom of the ladder is 3 m away from the wall.
(Continued)
Repeat your calculations, but for the rate at which the bottom end of the ladder is slide when the top of the ladder is just about to hit the ground.
Example: The ideal gas law states that

\[ PV = nRT \]

0.1 moles of gas are held in a piston, and the temperature is held constant at 273° K. The piston moves to decrease the volume from 3 l to 1 l over 60 seconds. What is the pressure half-way through this process, and at what rate is the pressure changing at this time?
(Continued)

\[ PV = nRT \]

\[ n = 0.1 \text{ mol, } T = 273 \, ^\circ\text{K} \]

\[ \frac{dL}{dt} = \frac{-1}{30} \, \text{l/s} \]
Example:  A radar tracking site is following a plane flying in a straight line. At its closest approach, the plane will be 150 km away from the radar site. If the plane is traveling at 600 km/h, how quickly is the radar rotating to track the plane when the plane is closest?
Does the angular rotation speed up or slow down as the plane moves away from the radar station?
Sketch a graph of the angular rotation rate against the rotation angle.