Unit #9: Definite Integral Properties, Fundamental Theorem of Calculus

Goals:

- Identify properties of definite integrals
- Define odd and even functions, and relationship to integral values
- Introduce the Fundamental Theorem of Calculus
- Compute simple anti-derivatives and definite integrals
Definite Integrals in Modeling

One of the primary applications of integration is to use a known rate of change, and compute the net change over some time interval.

**Example:** Suppose water is flowing into/out of a tank at a rate given by \( r(t) = 200 - 10t \) L/min, where positive values indicate the flow is into the tank.

Write an integral that expresses the change in the volume of water in the tank during the first 30 minutes of filling.

(a) \( \int_0^{30} (0 - 10) \, dt \)

(b) \( \int_0^{30} (200 - 10t) \, dt \)

(c) \( \int_0^{30} (200 - 10t) \Delta t \, dt \)

(d) \( \int_0^{30} \left( 200t - 10 \frac{t^2}{2} \right) \, dt \)
Estimate the integral using a left-hand rule with three intervals.

Does this information tell you the actual volume in the tank after 30 minutes? Why or why not?
**Question:** If \( h(t) \) represents the height of a child (in cm) at time \( t \) (in years), and the child is 120 cm tall at age 10, how would you represent the amount the child grew between \( t = 10 \) and \( t = 18 \) years?

A. \[ \int_{10}^{18} h(t) \, dt \]

B. \[ \int_{10}^{18} h(t) \, dt + 120 \]

C. \[ \int_{10}^{18} h'(t) \, dt \]

D. \[ \int_{10}^{18} h'(t) \, dt + 120 \]
Properties of Definite Integrals

Example: Sketch the area implicit in the integral \( \int_{-\pi/3}^{\pi/3} \cos(x) \, dx \)

If you were told that \( \int_{0}^{\pi/3} \cos(x) \, dx = \frac{\sqrt{3}}{2} \), find the size of the area you sketched.

(a) \( \int_{-\pi/3}^{\pi/3} \cos(x) \, dx = 0 \)
(b) \( \int_{-\pi/3}^{\pi/3} \cos(x) \, dx = 4 \frac{\sqrt{3}}{2} \)
(c) \( \int_{-\pi/3}^{\pi/3} \cos(x) \, dx = 2 \frac{\sqrt{3}}{2} \)
(d) \( \int_{-\pi/3}^{\pi/3} \cos(x) \, dx = \frac{\sqrt{6}}{2} \)
This example highlights an important and intuitive general property of definite integrals.

**Additive Interval Property of Definite Integrals**

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]

*Explain this general property in words and with a diagram.*
A less commonly used, but equally true, corollary of this property is a second property:

**Reversed Interval Property of Definite Integrals**

\[
\int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx
\]

Use the integral \( \int_{0}^{\pi/3} \cos(x) \, dx + \int_{\pi/3}^{0} \cos(x) \, dx \), and the earlier interval property, to illustrate the reversed interval property.
Give a rationale related to Riemann sums for the Reversed Interval property.
Even and Odd Functions

These properties can be helpful especially when dealing with even and odd functions.

*Define an even function. Give some examples and sketch them.*
Define an odd function. Give some examples and sketch them.
Integral Properties of Even and Odd Functions

Which of the following is a property specifically of odd functions when you integrate symmetrically over both sides of $x = 0$?

(a) \[ \int_{-a}^{a} f(x) \, dx = 2 \left( \int_{0}^{a} f(x) \, dx \right) \]

(b) \[ \int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx \]

(c) \[ \int_{-a}^{a} f(x) \, dx = 0 \]
Find a property of even functions when you integrate on both sides of $x = 0$.

(a) $\int_{-a}^{a} f(x) \, dx = 2 \left( \int_{0}^{a} f(x) \, dx \right)$

(b) $\int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx$

(c) $\int_{-a}^{a} f(x) \, dx = 0$
Linearity of Definite Integrals

Example: If $\int_{a}^{b} f(x) \, dx = 10$, then what is the value of $\int_{a}^{b} 5f(x) \, dx$? Sketch an area rationale for this relation.
Example: If $\int_a^b f(x) \, dx = 2$, and $\int_a^b g(x) \, dx = 4$ then what is the value of $\int_a^b [f(x) + g(x)] \, dx$? Again, sketch an area rationale for this relation.
Linearity of Definite Integrals

\[ \int_{a}^{b} k f(x) \, dx = k \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \]

\[ \int_{a}^{b} (f(x) - g(x)) \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx \]
Simple Bounds on Definite Integrals

Example: Sketch a graph of \( f(x) = 5 \sin(2\pi x) \), then use it to make an area argument proving the statement that

\[
0 \leq \int_{0}^{\frac{1}{2}} 5 \sin(2\pi x) \, dx \leq \frac{5}{2}
\]
Simple Maximum and Minimum Values for Definite Integrals
If a function \( f(x) \) is continuous and bounded between \( y = m \) and \( y = M \) on the interval \([a, b]\), i.e. \( m \leq f(x) \leq M \) on the interval, then

\[
m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)
\]
Note that the maximum and minimum values we get with the method above are quite crude. Sometimes you will be asked for much more precise values which can often be just as easy to find.

Example: Use a graph to find the exact value of \( \int_{0}^{1} 5 \sin(2\pi x) \, dx \). I.e. not just a range, but the single correct integral value.

(a) \( \int_{-a}^{a} 5 \sin(2\pi x) \, dx = 5 \)

(b) \( \int_{-a}^{a} 5 \sin(2\pi x) \, dx = 5(2\pi) \)

(c) \( \int_{-a}^{a} 5 \sin(2\pi x) \, dx = \frac{5}{2\pi} \)

(d) \( \int_{-a}^{a} 5 \sin(2\pi x) \, dx = 0 \)
Relative Sizes of Definite Integrals

Example: Two cars start at the same time from the same starting point. For the first second,

- the first car moves at velocity $v_1 = t$, and
- the second car moves at velocity $v_2 = t^2$.

Sketch both velocities over the relevant interval.
Which car travels further in the first second? Relate this to a definite integral.

**Comparison of Definite Integrals**

If \( f(x) \leq g(x) \) on an interval \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} g(x) \, dx
\]
The Fundamental Theorem of Calculus

We have now drawn a firm relationship between area calculations (and physical properties that can be tied to an area calculation on a graph). The time has now come to build a method to compute these areas in a systematic way.

The Fundamental Theorem of Calculus

If $f$ is continuous on the interval $[a, b]$, and we define a related function $F(x)$ such that $F'(x) = f(x)$, then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$
The fundamental theorem ties the *area* calculation of a definite integral back to our earlier *slope* calculations from derivatives. However, it changes the direction in which we take the derivative:

- Given $f(x)$, we find the *slope* by finding the *derivative* of $f(x)$, or $f'(x)$.

- Given $f(x)$, we find the *area* $\int_a^b f(x) \, dx$ by finding $F(x)$ which is the *anti-derivative* of $f(x)$; i.e. a function $F(x)$ for which $F'(x) = f(x)$. 
In other words, if we can find an anti-derivative $F(x)$, then calculating the value of the definite integral requires a simple evaluation of $F(x)$ at two points ($F(b) - F(a)$). This last step is *much* easier than computing an area using finite Riemann sums, and also provides an exact value of the integral instead of an estimate.
Example: Use the Fundamental Theorem of Calculus to find the area bounded by the x-axis, the line $x = 2$, and the graph $y = x^2$. Use the fact that $\frac{d}{dx} \left( \frac{1}{3} x^3 \right) = x^2$. 
We used the fact that $F(x) = \frac{1}{3}x^3$ is an anti-derivative of $x^2$, so we were able to use the Fundamental Theorem. Give another function $F(x)$ which would also satisfy $\frac{d}{dx} F(x) = x^2$.

Use the Fundamental Theorem again with this new function to find the area implied by $\int_0^2 x^2 \, dx$. 
Did the area/definite integral value change? Why or why not?

Based on that result, give the most general version of $F(x)$ you can think of.

Confirm that you still satisfy $\frac{d}{dx} F(x) = x^2$. 
With our extensive practice with derivatives earlier, we should find it straightforward to determine some simple anti-derivatives. 

*Complete the following table of anti-derivatives.*

<table>
<thead>
<tr>
<th>function $f(x)$</th>
<th>anti-derivative $F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>$\frac{x^3}{3} + C$</td>
</tr>
<tr>
<td>$x^n$</td>
<td></td>
</tr>
<tr>
<td>$x^2 + 3x - 2$</td>
<td></td>
</tr>
<tr>
<td>function $f(x)$</td>
<td>anti-derivative $F(x)$</td>
</tr>
<tr>
<td>----------------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>$\cos x$</td>
<td></td>
</tr>
<tr>
<td>$\sin x$</td>
<td></td>
</tr>
<tr>
<td>$x + \sin x$</td>
<td></td>
</tr>
<tr>
<td>function $f(x)$</td>
<td>anti-derivative $F(x)$</td>
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<tr>
<td>-----------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>$e^x$</td>
<td></td>
</tr>
<tr>
<td>$2^x$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1 - x^2}}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{1 + x^2}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td></td>
</tr>
</tbody>
</table>
The chief importance of the **Fundamental Theorem of Calculus (F.T.C.)** is that it enables us (potentially at least) to find values of definite integrals more accurately and more simply than by the method of calculating Riemann sums. In principle, the F.T.C. gives a precise answer to the integral, while calculating a (finite) Riemann sum gives you no better than an approximation.
Example: Consider the area of the triangle bounded by $y = 4x$, $x = 0$ and $x = 4$. Compute the area based on a sketch, and then by constructing an integral and using anti-derivatives to compute its value.
Example: Use a definite integral and anti-derivatives to compute the area under the parabola $y = 6x^2$ between $x = 0$ and $x = 5$. 
Basic Anti-Derivatives - $1/x$

The last entry in our anti-derivative table was $f(x) = \frac{1}{x}$. It is a bit of a special case, as we can see in the following example.

**Example:** Sketch the region implied by the integral $\int_{-3}^{-1} \frac{1}{x} \, dx$. 
Example: Now use the anti-derivative and the Fundamental Theorem of Calculus to obtain the exact area under $f(x) = \frac{1}{x}$ between $x = -3$ and $x = -1$. Make any necessary adaptations to our earlier anti-derivative table.
Basic Anti-Derivatives - $1/x - 3$
Anti-derivatives and the Fundamental Theorem of Calculus

The F.T.C. tells us that if we want to evaluate

$$\int_{a}^{b} f(x) \, dx$$

all we need to do is find an anti-derivative $F(x)$ of $f(x)$ and then evaluate $F(b) - F(a)$.

THERE IS A CATCH. While in many cases this really is very clever and straightforward, in other cases finding the anti-derivative can be surprisingly difficult. This week, we will stick with simple anti-derivatives; in later weeks we will develop techniques to find more complicated anti-derivatives.

Some general remarks at this point will be helpful.
Remark 1

Because of the importance of finding an anti-derivative of $f(x)$ when you want to calculate $\int_a^b f(x) \, dx$, it has become customary to denote the anti-derivative itself by the symbol

$$\int f(x) \, dx$$

The symbol $\int f(x) \, dx$ (with no limits on the integral) refers to the anti-derivative(s) of $f(x)$, and is called the **indefinite integral** of $f(x)$.

Note that the definite integral is a number, but the indefinite integral is a function (really a family of functions).
Remark 2

Since there are always infinitely many anti-derivatives, all differing from each other by a constant, we customarily write the anti-derivative as a family of functions, in the form $F(x) + C$. For example,

$$\int x^2 \, dx = \frac{x^3}{3} + C$$

Note that an anti-derivative is a single function, while the indefinite integral is a family of functions.
Remark 3

Since the last step in the evaluation of the integral \( \int_a^b f(x) \, dx \), once the antiderivative \( F(x) \) is found, is the evaluation \( F(b) - F(a) \), it is customary to write \( F(x) \bigg|_a^b \) in place of \( F(b) - F(a) \), as in

\[
\int_0^4 x^2 \, dx = \frac{x^3}{3} \bigg|_0^4 = \frac{4^3}{3} - \frac{0^3}{3}
\]

Remark 4

The variable \( x \) in the definite integral \( \int_a^b f(x) \, dx \) is called the *variable of integration*. It can be replaced by another variable name without altering the value of the integral.

\[
\int_a^b f(x) \, dx = \int_a^b f(u) \, du = \int_a^b f(\theta) \, d\theta
\]