Unit #14 : Center of Mass, Improper Integrals

Goals:

- Apply the slicing integral approach to computing more complex totals calculations, including center of mass.
- Learn how to evaluate integrals involving infinite quantities.

In earlier units we reviewed how to compute the total area or volume of complex shapes by taking slices of the shape. This week we will study more abstract applications of the same idea.

Example: *A metal rod has length two meters. At a distance of x meters from its left end, the density of the rod is given by $\delta(x) = 3 + 0.5x$ kg per meter.*

Why can't we simply compute the total mass using

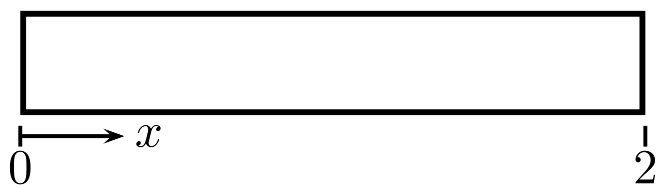
$$\textit{mass (kg)} = [\textit{lineal density (kg/m)}] \times [\textit{ length (m)}]$$

(a) The density isn't just one value.

(b) The length isn't just one value.

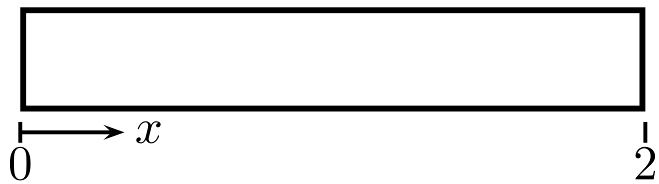
(c) The units of density aren't correct.

Breaking the rod down into small segments Δx *long, construct a Riemann sum for the total mass of the metal rod.*



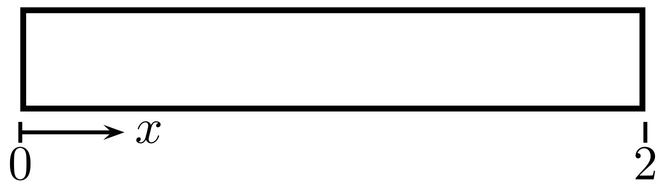
$$\delta(x) = 3 + 0.5x$$

Find the exact mass of the rod by converting the Riemann sum to an integral and evaluating it.



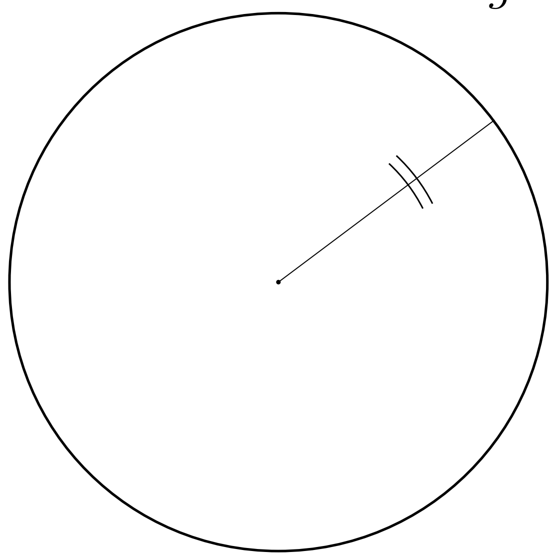
$$\delta(x) = 3 + 0.5x$$

Can you check that the answer you obtained is reasonable using a quick alternative approximation? Do so.

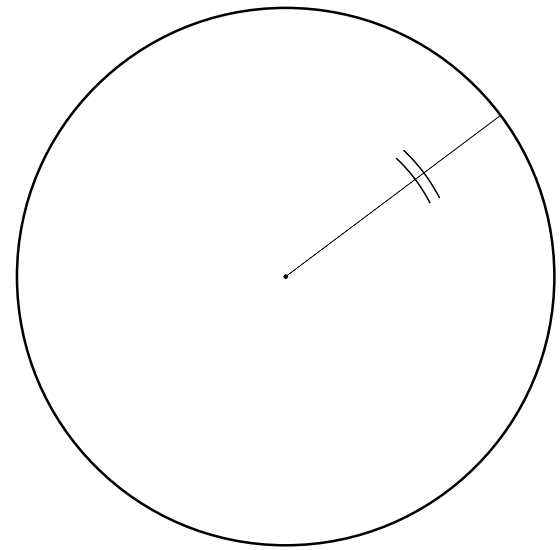


$$\delta(x) = 3 + 0.5x$$

Example: *The density of oil in a circular oil slick on the surface of the ocean, at a distance of r meters from the center of the slick, is given by $\delta(r) = \dfrac{100 - r^2}{10}$ kg/m^2. We want to find the total mass of the oil in the slick.*
Consider a ring of radius r and thickness dr concentric with the oil slick. Write down the expression for the mass of the oil contained within that ring.



Write down and evaluate the integral that represents the total mass of the oil slick. (Think about what outer radius we should use.)



Center of Mass

A property of an object related to density and mass is the **center of mass**. It can be thought of as the *balance point* of the object or system.
In general, when dealing with a set of point masses m_i at locations x_i, their center of mass is given by

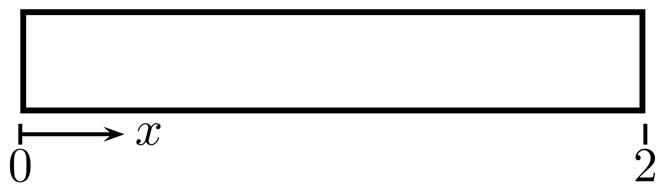
$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}$$

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}$$

Example: *A mother and daughter sit on opposite ends of a 3 m long see-saw; the mother has a mass of 60 kg, while her daughter weighs 20 kg. How close to the mother's end would the support need to be put so that the two of them would be in perfect balance?*

Unlike the last question, many problems in physics and chemistry involve mass spread out in a more continuous way.

Example: *Consider the center of mass for the metal rod we studied earlier: two meters long, with density* $\delta(x) = 3 + 0.5x$ *kg/m.*



$\delta(x) = 3 + 0.5x$

Based on the balance-point interpretation of the center of mass, where will the center of mass for this rod be?

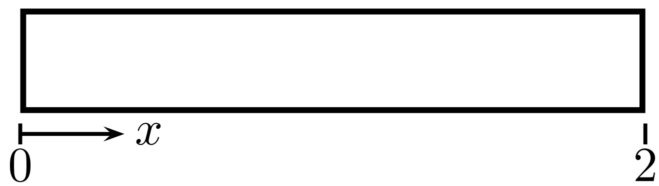
(a) To the left of $x = 1$.

(b) Exactly at $x = 1$.

(c) To the right of $x = 1$.

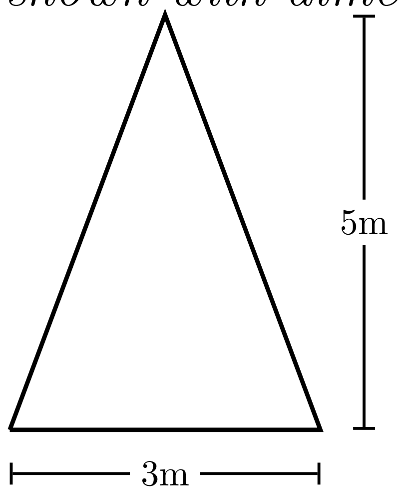
$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}$$

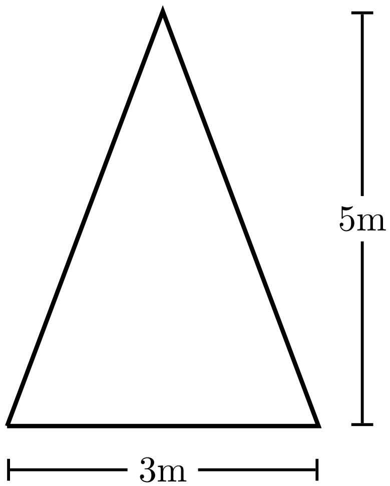
Extend the discrete formula for center of mass (above) to a continous model by using a slicing approach. Use this to find the exact center of mass of the rod.



$$\delta(x) = 3 + 0.5x$$

Example: *Find the center of mass for a triangular piece of sheet metal shown with dimensions shown below.*




5m
3m

Improper Integrals

A common type of integral that appears naturally in many sciences and economics is an integral with an **infinite limit**. This can force us to face some slippery ideas about infinity...

Example: *A chemical reaction produces a desired chemical at a rate of* $r(t) = e^{-t}$ *g/s. What amount is produced between* $t = 0$ *and some unspecified later time,* $t = T$*?*

How would you answer the question "What is the total amount of chemical that this reaction could produce, if it were run forever?"

We can write the answer to this question as an **improper integral.**

Improper Integrals

Improper integrals are integrals which somehow involve an **infinite quantity**. Both are computed using **limits**.

- Infinite limit of integration:

$$\underbrace{\int_a^{\infty} f(t)\,dt}_{\text{usual form}} = \underbrace{\lim_{T\to\infty}\int_a^{T} f(t)\,dt}_{\text{exact definition}}$$

- Infinity in the integrand:

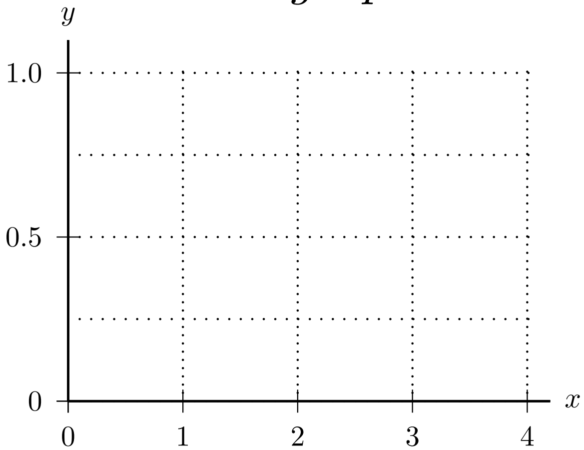
$$\text{if } f(a) \text{ is undefined, } \int_a^b f(t)\,dt = \lim_{T\to a}\int_T^b f(t)\,dt$$

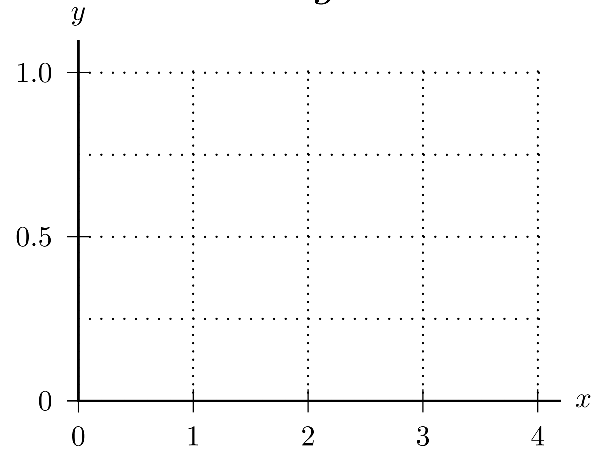
If we can evaluate the limit and obtain a finite number, we say **the integral converges**. If the limit does not exist, we say the integral **diverges**.

Example: *Use the definition of improper integrals to evaluate* $\int_1^{\infty} \frac{1}{t^2}\, dt$

Example: *Evaluate* $\int_1^\infty \frac{1}{t}\,dt$

Sketch both graphs and indicate what area is being evaluated in each case.





How much does graph sketching help in analyzing these types of integrals?

(a) Graph sketching helps evaluating these integrals .

(b) Graph sketching does not help evaluate these integrals.

Families of functions

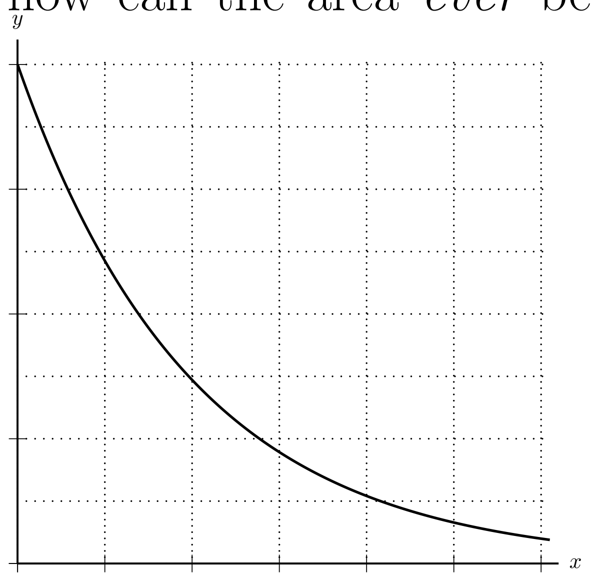
Show that all the integrals $\int_1^\infty e^{kt}\,dt$ *converge for* $k < 0$.

Show that all the integrals $\displaystyle\int_1^\infty \frac{1}{t^k}\, dt$ *converge for* $k > 1$.

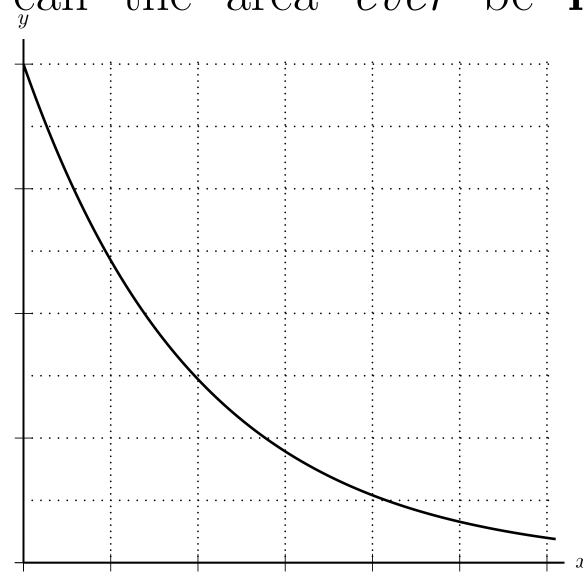
Reasonableness of Infinite Integrals

Students often approach infinite integrals initially from one of two stances:

(a) If we integrate out to infinity, how can the area *ever* be **finite**?



(b) If the function we're integrating is going to zero, how can the area *ever* be **infinite**?



Both of these ideas reflect the struggle about the role of infinity and zero in the integral.

Can we ever have an infinitely long object that only has finite area?

Can we ever have a function that goes to zero, but has infinite area underneath it?

Common Family Concept Check

From our earlier analysis of common families of functions, we can say that for an integral up to infinity to converge,

- **the integrand must to go zero** and
- **it must go to zero faster than** $\frac{1}{t}$**.**

Question: Does $\int_1^\infty e^{-0.01t}\, dt$ converge / is the area it represents finite?

1. Yes

2. No

Question: Does $\int_1^\infty \frac{1}{t^2}\, dt$ converge / is the area it represents finite?

1. Yes

2. No

Question: Does $\int_1^{\infty} \frac{1}{\sqrt{t}}\, dt$ converge / is the area it represents finite?

1. Yes

2. No

Infinite Integrands

Sometimes, an integral involves an *implicit* infinity e.g. when the integrand (function) goes to infinity at the edge of our interval. Fortunately, we can use a similar limit approach to evaluate this case as well. Again, these integrals may end up being infinite or not, depending on how quickly the function approaches infinity.

Example: *Evaluate the integral* $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Example: *Evaluate the integral* $\int_0^1 \frac{1}{x^3}\, dx$

Applications of Improper Integrals: Universal Law of Gravitation

The Universal Law of Gravitation gives the force of attraction between two masses m_1 and m_2 (in kilograms) which are a distance of r meters apart by the formula

$$F = \frac{Gm_1m_2}{r^2} \quad ,$$

where $G = 0.667 \times 10^{-10}$ is the gravitational constant and F is measured in newtons.

This formula is especially interesting when m_2 is the mass of the Earth and m_1 is the mass of an object in its gravitational field. For objects close to the Earth's surface, $r \approx r_0 = 6.38 \times 10^6$ meters, the radius of the Earth.

When an object of mass m_1 is lifted away from the Earth's surface by a small distance d meters (say a math prof going up two stories in an elevator), then the amount of work done by the elevator is equal to force times distance; that is,

$$\text{Work} = Fd = \frac{Gm_1m_2}{r_0^2} \cdot d \ ,$$

with the result measured in joules (J). On the other hand, to put a satellite in an orbit at a height r meters from the Earth's center, we have to account for the fact that force decreases as we move away from the Earth. This means that the **total work done** will be represented by an integral rather than simply the product $F(r - r_0)$.

Write down the integral that represents the work required to move a mass of m_1 kg from the surface of the earth, r_0 from the center, to a point completely out of the reach of Earth's gravity.

Evaluate the integral you found, leaving the constants in as letters.

Sub in the following values based on lifting an object from the surface of the earth:

- $m_1 = 70$ kg, the rough mass of a person
- $m_2 = 6 \times 10^{24}$ kg as the mass of the earth
- $G = 6.67 \times 10^{-11}$ N m^2/kg^2 is the universal gravitation constant
- $r_0 = 6.38 \times 10^6$ m is the distance from the center to the Earth's surface

To escape the Earth's pull without further assistance, a rocket must be moving fast enough so that its kinetic energy while moving at the Earth's surface is equal to the amount of energy we just found. If kinetic energy of an object moving at velocity v is given by $E = \frac{1}{2}mv^2$, how quickly must an object at the surface of the earth be moving escape the Earth's gravitational pull completely? (This speed is know as the **escape velocity** *for the Earth.)*