

Unit #18 : Level Curves, Partial Derivatives

Goals:

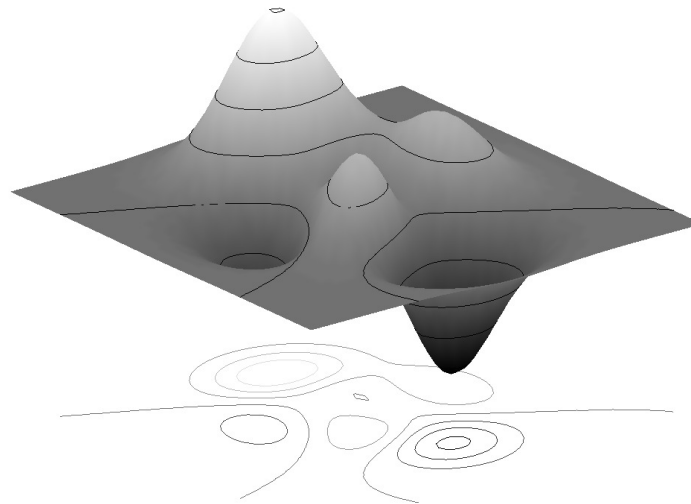
- To learn how to use and interpret contour diagrams as a way of visualizing functions of two variables.
- To study linear functions of two variables.
- To introduce the partial derivative.

Picturing $f(x, y)$: Contour Diagrams (Level Curves)

We saw earlier how to sketch surfaces in three dimensions. However, this is not always easy to do, or to interpret. A contour diagram is a second option for picturing a function of two variables.

Suppose a function $h(x, y)$ gives the height above sea level at the point (x, y) on a map. Then, the graph of h would resemble the actual landscape.

Suppose the function h looks like this:



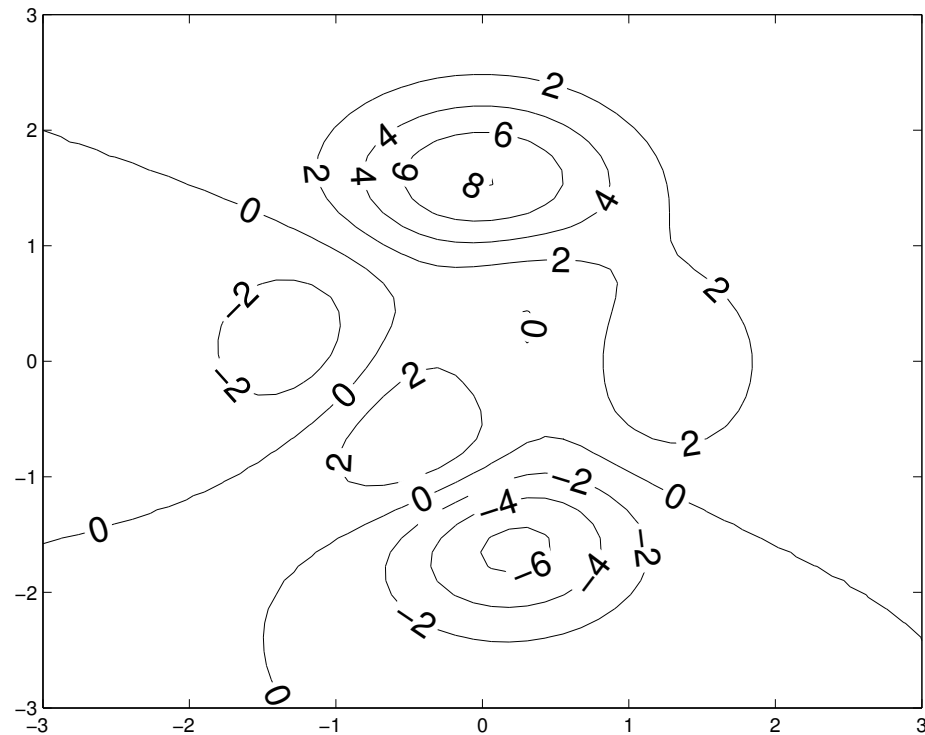
Then, the **contour diagram** of the function h is a picture in the (x, y) -plane showing the **contours**, or **level curves**, connecting all the points at which h has the same value. Thus, the equation

$$h(x, y) = 100$$

gives all the points where the function value is 100.

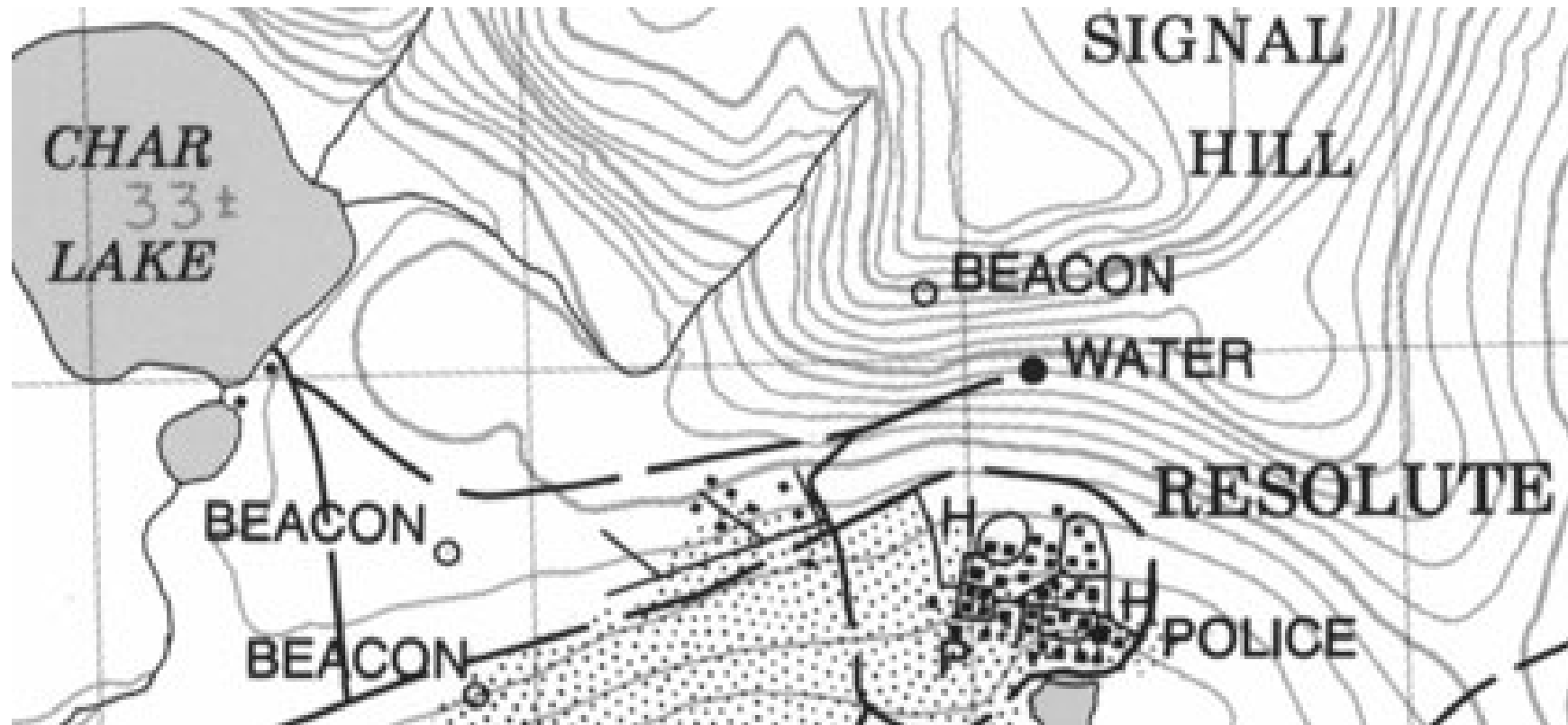
Together they usually constitute a curve or a set of curves called the contour or level curve for that value. In principle, there is a contour through every point. In practice, just a few of them are shown.

The following is the contour diagram for the earlier surface.



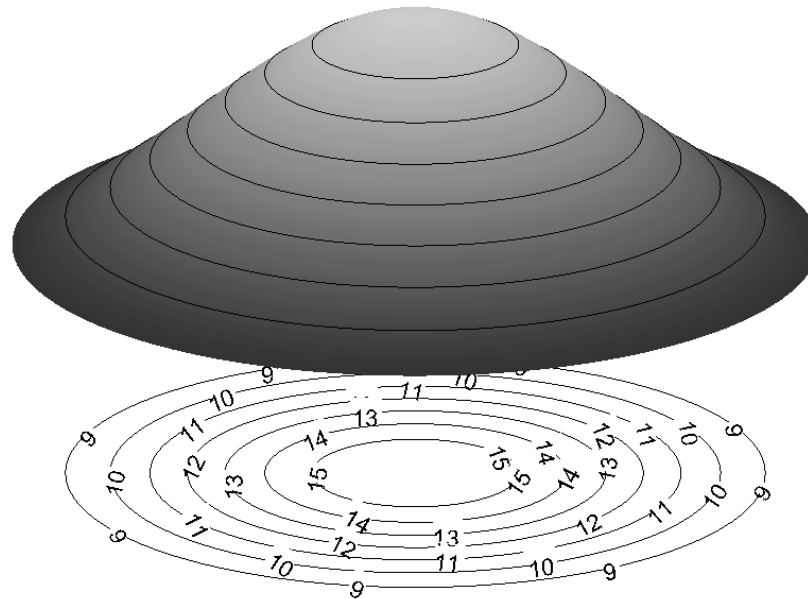
Indicate the location of the peaks and pits/valleys on the contour diagram.

Topographic maps are also contour maps.



Identify first a steep path, and then a more flat path, from the town up to Signal hill.

In principle, the contour diagram and the graph can each be reconstructed from the other. Here is a picture illustrating this:



As shown above, the contour $f(x, y) = k$ is obtained by intersecting the graph of f with the horizontal plane, $z = k$, and then dropping (or raising) the resulting curve to the (x, y) -plane. The graph is obtained by raising (or dropping) the contour $f(x, y) = k$ to the level $z = k$.

Interpreting Contour Diagrams

Match each of the following functions to their corresponding contour diagram.

(1) $h(x, y)$ is the degree of pleasure you get from a cup of coffee when

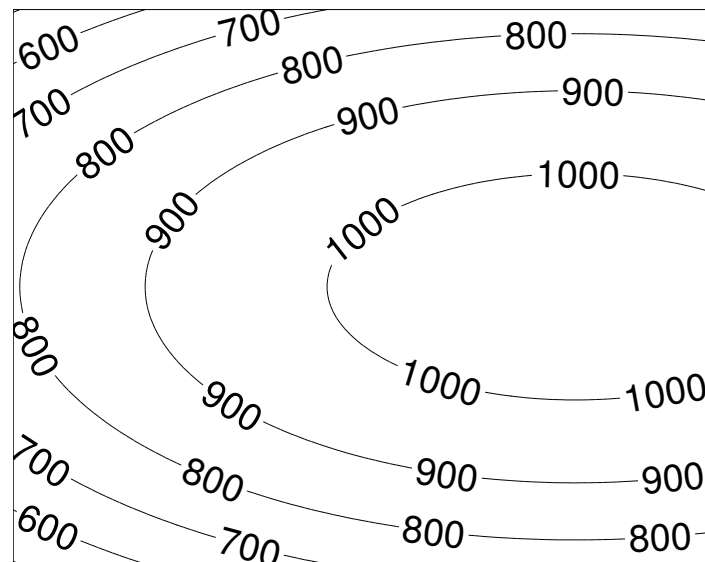
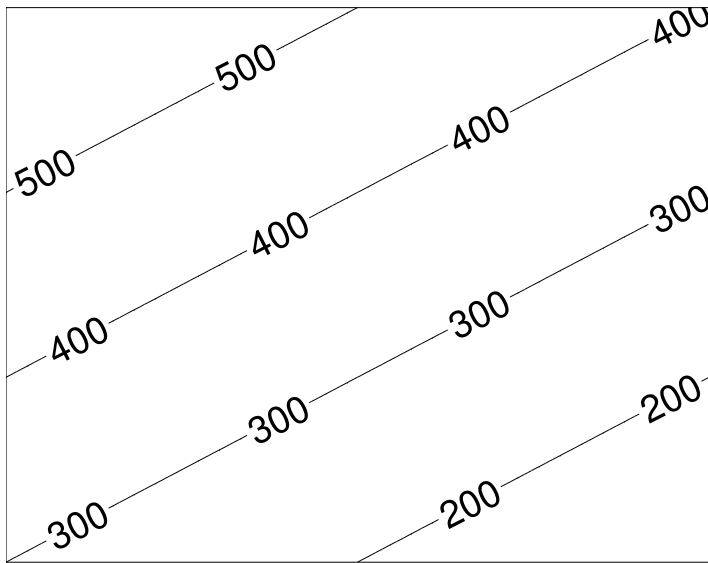
- x is the temperature, and
- y is the amount of ground coffee used to brew it.

(2) $f(x, y)$ is the number of TV sets sold when

- x is the price per TV set, and
- y is the amount of money spent weekly on advertising.

(3) $g(x, y)$ is the amount of gas per week sold by a gas station when

- x is the amount spent on bonus gifts to customers, and
- y is the price charged by a nearby competitor.



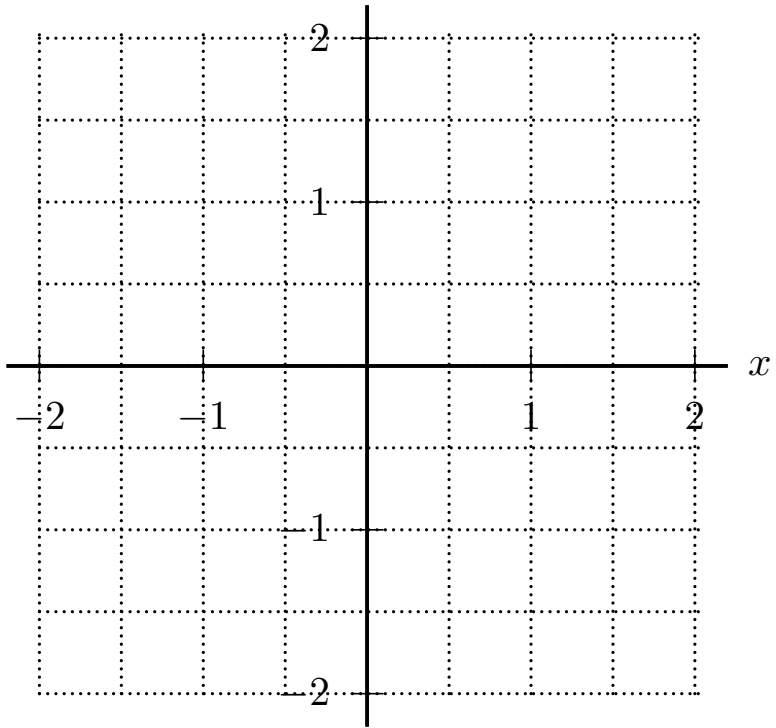
Useful Properties of Contour Plots

If the contour lines are evenly spaced in their z values,

- contour lines closer together indicate more rapid change/steeper slopes.
- contour lines further apart indicate flatter regions.
- peaks and valleys look the same; only the values of the contours let you distinguish them.

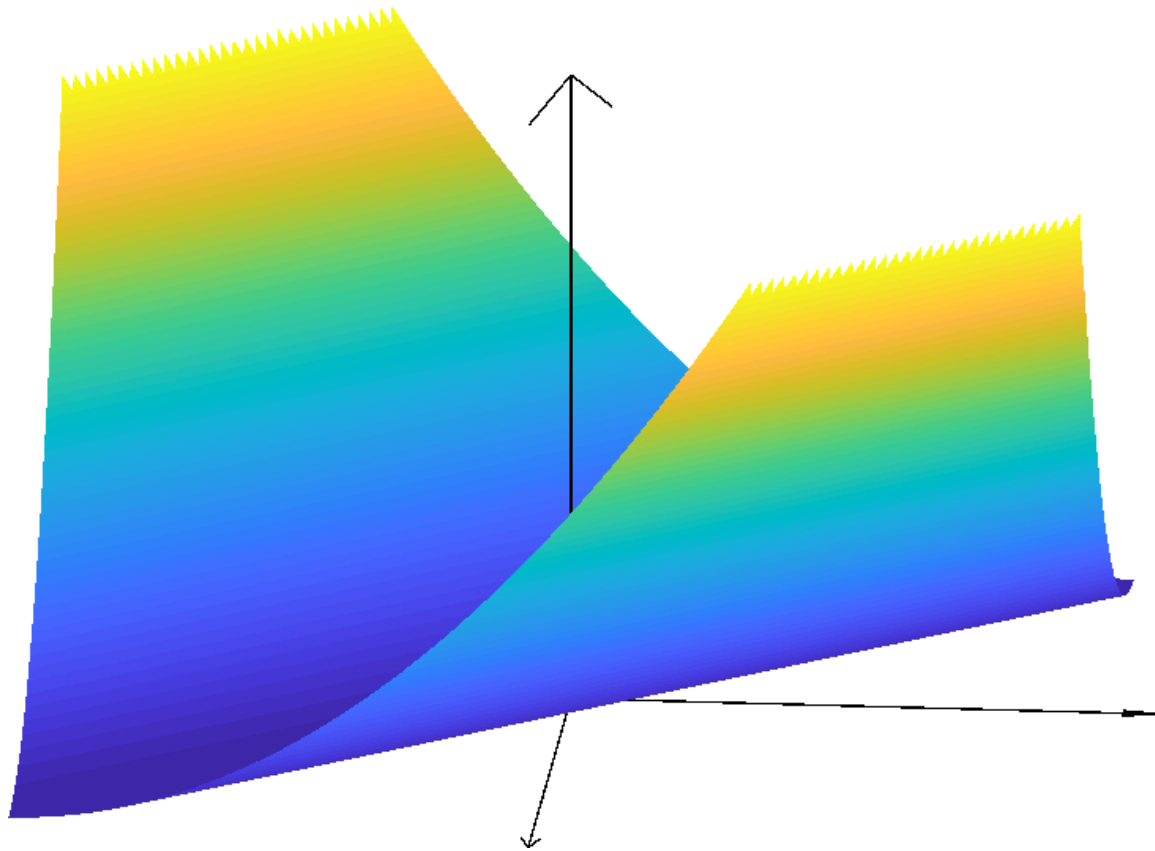
Example

Draw the contours of $f(x, y) = (x + y)^2$ for the values $f = 1, 2, 3,$ and 4 .

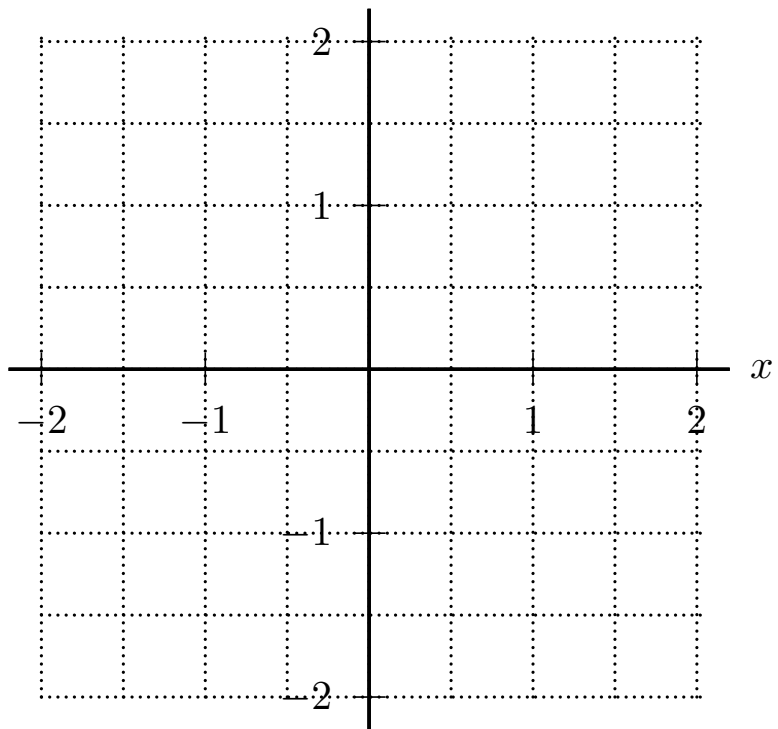


Give a verbal description of the surface defined by $f(x, y) = (x + y)^2$.

Here is the function $f(x, y) = (x + y)^2$ plotted as a surface.



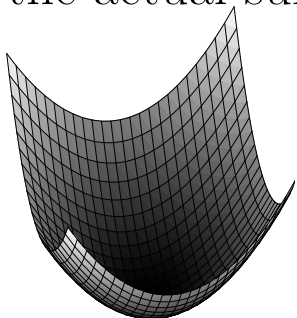
Example: Draw the contours of $f(x, y) = x^2 + y^2$ for the values 1, 2, 3, and 4.



From the contour diagram, is the value of f at $(0,0)$ a local minimum or maximum?

Is the surface becoming more or less steep as you move away from the origin?

Try to associate how the lines on the contour diagram could help you to imagine the actual surface:



Linear Functions of Two Variables

A function of two variables is **linear** if its formula has the form

$$f(x, y) = c + mx + ny.$$

The textbook shows that m and n can be interpreted as **slopes in the x -direction and the y -direction**, respectively, and that c is **the z -intercept**.

Consider the plane $z = 2 - x - y$. This plane has slope -1 in both the x - and the y -directions. Is there any direction in which it has a steeper slope?

It may help to experiment by holding up a book or other flat object.

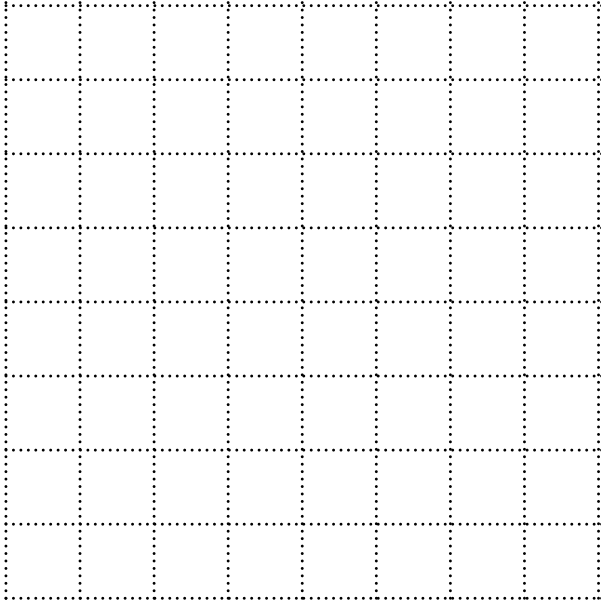
Linear Example

Given that the following is a table of values for a linear function, f , complete the table.

		x			
		2	3	4	5
y	1.1	2			
	1.2				
	1.3			21	
	1.4	8			

Give a formula for the function $f(x, y)$ in the preceding example.

Sketch contours of the linear function f .



Based on this example, hypothesize properties of the contour diagrams for all linear functions.

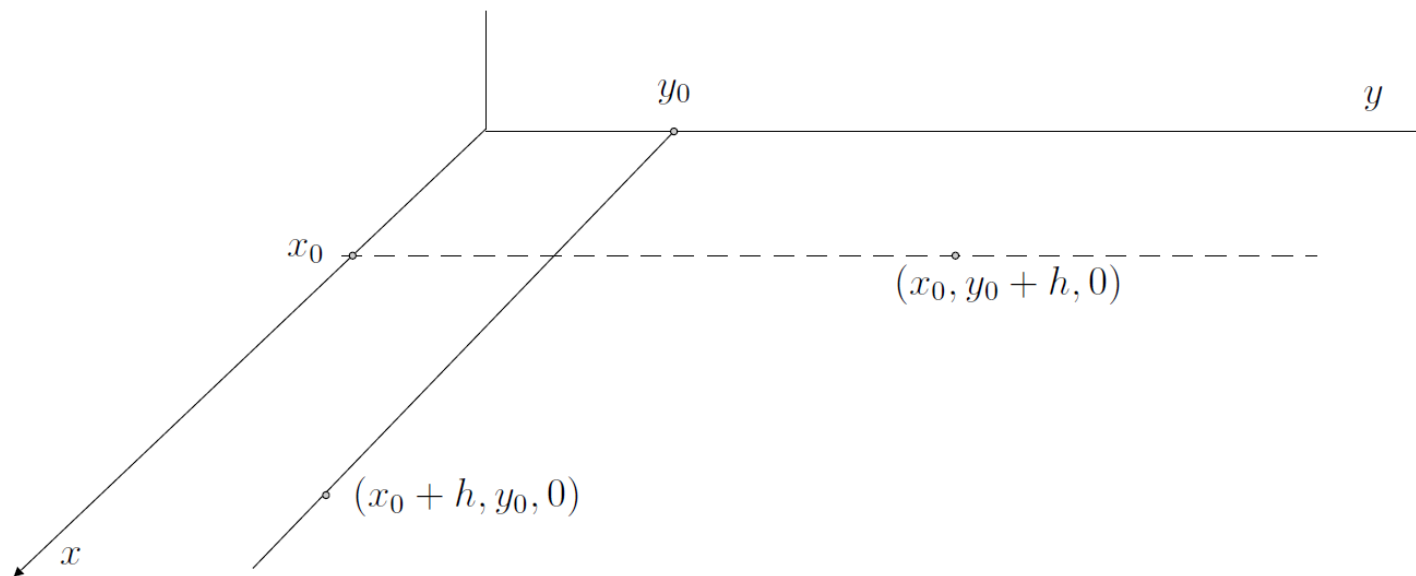
Partial Derivatives

Just as $\frac{df}{dx}$ is the rate of change of $f(x)$ when x is changed, so the derivatives of $f(x, y)$ are the rates of change of the function value when **one** of the variables is changed. Since there are two variables to choose from, there are two derivatives, one to describe what happens when you change x **only** and one to describe what happens when you change y **only**. Because either derivative by itself describes the behaviour of the function only **partly**, they are called **partial derivatives**.

If we look at a graph of $z = f(x, y)$, partial derivatives tell us how the height of the graph, z , is changing as the point $(x, y, 0)$ moves along a line parallel to the x -axis or parallel to the y -axis.

$(x_0, y_0, 0) \longrightarrow (x_0 + h, y_0, 0)$ (along solid line parallel to the x -axis).

$(x_0, y_0, 0) \longrightarrow (x_0, y_0 + h, 0)$ (along dotted line parallel to the y -axis).



Definition of Partial Derivatives

$$\begin{aligned}\frac{\partial f}{\partial x}(x_0, y_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\ &= f_x(x_0, y_0)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}(x_0, y_0) &= \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \\ &= f_y(x_0, y_0)\end{aligned}$$

As a shorthand,

- $f_x(x_0, y_0)$ also means the same as $\frac{\partial f}{\partial x}(x_0, y_0)$, and
- $f_y(x_0, y_0)$ also means the same as $\frac{\partial f}{\partial y}(x_0, y_0)$.

There is no $f'(x, y)$ notation possible when working with multivariate functions.

To actually calculate $\frac{\partial f}{\partial x}$ when we have a formula for $f(x, y)$, we imagine that y is fixed, then we have a function of only ONE variable and we take its derivative in the usual way. To calculate $\frac{\partial f}{\partial y}$, we imagine that x is fixed and y is not.

Example: Consider $f(x, y) = x^2y + \sin(xy)$.

Write a new single-variable function: $g(x) = f(x, y_0)$

Find $\frac{dg}{dx}$.

More practically, when we are looking for $\frac{\partial f}{\partial x}(x, y)$ — a formula in terms of x and y — we do not actually replace y by y_0 , but simply *think of it* as a constant.

$$f(x, y) = x^2y + \sin(xy)$$

Find $\frac{\partial f}{\partial x}(x, y)$.

Find $\frac{\partial f}{\partial y}(x, y)$.

Partial Derivative Practice

Example: *Find both partial derivatives for $f(x, y) = (1 + x^3)y^2$.*

Example: Find both partial derivatives for $f(x, y) = e^x \sin(y)$.

Example: Find both partial derivatives for $f(x, y) = \frac{x^2}{4y}$.

Question: $\frac{\partial}{\partial y} (x^2 \tan(y) + y^2 + x)$ is

(a) $2x \sec^2(y) + 2y + 1$

(b) $x^2 \sec^2(y) + 2y$

(c) $2x \tan(y) + 1$

(d) $x^2 \tan(y) + y^2 + 1$

Example: Consider $f(x, y) = x^2 \sin y + e^{xy^2}$.

Find the value of $\frac{\partial f}{\partial x}(1, 0)$.

Find the value of $\frac{\partial f}{\partial y}(1, 0)$.

What do these values partial derivative values tell you about the graph of $f(x, y) = x^2 \sin y + e^{xy^2}$ near $(1, 0)$?

Partial Derivatives - Economics

Example: What are the signs of $\frac{\partial g}{\partial x}(x, y)$ and $\frac{\partial g}{\partial y}(x, y)$ if

- $g(x, y)$ is the amount of gas sold per week by a gas station,
- x is the station's price for gas, and
- y is the price charged by a nearby competitor?

(a) $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are both positive.

(b) $\frac{\partial g}{\partial x}$ is positive, $\frac{\partial g}{\partial y}$ is negative.

(c) $\frac{\partial g}{\partial x}$ is negative, $\frac{\partial g}{\partial y}$ is positive.

(d) $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are both negative.

Example: What would you expect the signs of $\frac{\partial h}{\partial x}(x, y)$ and $\frac{\partial h}{\partial y}(x, y)$ to be if

- $h(x, y)$ is the number of pairs of ski lift tickets sold in a year in Canada,
- x is the number of ski boots sold, and
- y is the number of tickets from Canada to warmer vacation spots.

(a) $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are both positive.

(b) $\frac{\partial h}{\partial x}$ is positive, $\frac{\partial h}{\partial y}$ is negative.

(c) $\frac{\partial h}{\partial x}$ is negative, $\frac{\partial h}{\partial y}$ is positive.

(d) $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are both negative.

Economics key words: *substitutes* and *complements*.

Partial Derivatives - Ideal Gas Law

Recall the ideal gas law, written with pressure as a function of temperature and volume:

$$P(V, T) = \frac{nRT}{V}$$

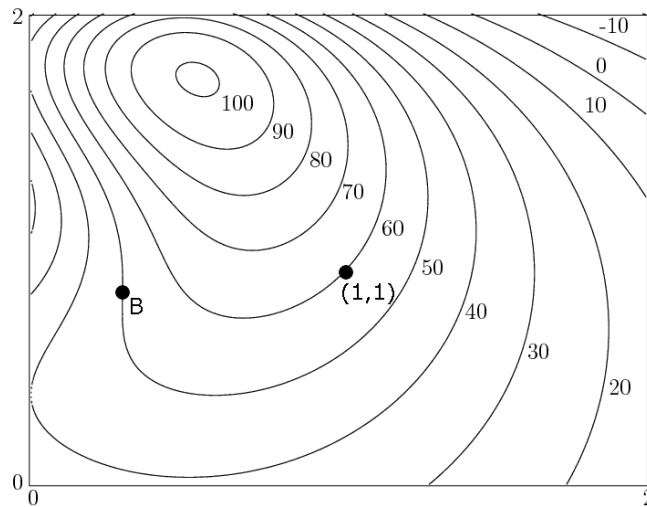
For one mole of gas ($n = 1$) and the SI units of kPa, liters and degrees Kelvin, find $\frac{\partial P}{\partial T}$ and $\frac{\partial P}{\partial V}$. Give the units of both derivatives.

*Evaluate both derivatives at $T = 300^\circ K$ and $V = 10$ liters. Use $R = 8.31 L$
 $kPa / (K mol)$*

Express the meaning of both $P_T(10 \text{ L}, 300^\circ \text{ K})$ and $P_V(10 \text{ L}, 300^\circ \text{ K})$ in words.

Partial Derivatives from Contour Diagrams

Example: Consider the contour diagram shown below, representing the function $h(x, y)$.



Question: $\frac{\partial h}{\partial x}(1, 1)$ is
 (a) positive

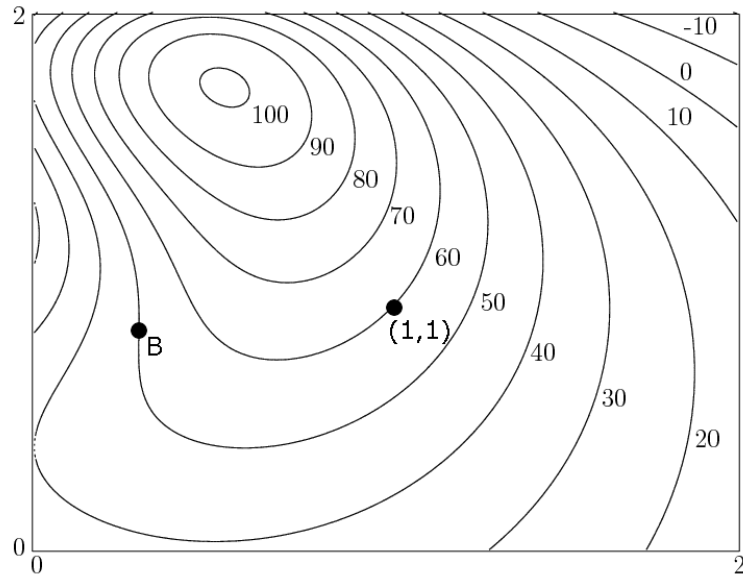
(b) negative

(c) zero

Question: $\frac{\partial h}{\partial y}(1, 1)$ is
 (a) positive

(b) negative

(c) zero



Question: $\frac{\partial h}{\partial x}$ at the point B is

(a) positive

(b) negative

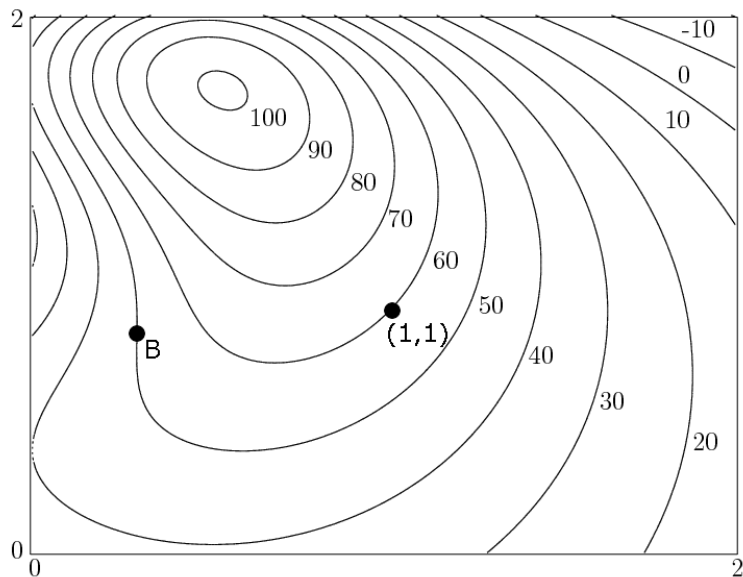
(c) zero

Question: $\frac{\partial h}{\partial y}$ at the point B is

(a) positive

(b) negative

(c) zero



On the same contour diagram, mark a point where $\frac{\partial h}{\partial x}$ seems particularly large.

Partial Derivatives from Table Data

Given the following table of values for $f(x, y)$, calculate **approximate** values for $\frac{\partial f}{\partial x}(1, 1)$ and $\frac{\partial f}{\partial y}(1, 1)$. (Multiple answers are possible, because we are estimating.)

		x			
		0.9	1.0	1.1	1.2
y	0.9	6.62	5.47	4.44	3.53
	1.0	8.93	7.39	5.99	4.75
	1.1	11.13	9.97	8.08	6.42
	1.2	16.28	13.46	10.91	8.67

What do the values $f_x(1, 1)$ and $f_y(1, 1)$ tell you about the graph of $f(x, y)$ near $(1, 1)$?