The Division Algorithm

Definition: Let $f, g \in R[x]$ be polynomials. Then we say that f divides g in $R[x]$, if there is a polynomial $h \in R[x]$ such that

$$g = f \cdot h;$$

we then write $f|_R g$ (or just $f|g$, if the reference to R is clear).

Theorem 2 (Division algorithm for $F[x]$)
Let $F = \mathbb{C}, \mathbb{R}, \mathbb{Q}$ or $\mathbb{F}_p$ (but not $\mathbb{Z}$!). Then for each pair $f, g \in F[x]$, $g \neq 0$, there exist unique polynomials $q, r \in F[x]$ such that

$$f(x) = q(x)g(x) + r(x), \qquad (1)$$

$$\deg(r) < \deg(g). \qquad (2)$$

Notation: We write
$\mathrm{quot}(f, g) := q(x)$, the quotient of f by g,
$\mathrm{rem}(f, g) := r(x)$, the remainder of f by g.

Corollary 1: $g|_F f \iff \mathrm{rem}(f, g) = 0.$

Corollary 2: If $f, g \in \mathbb{Q}[x]$, then

$$f|_{\mathbb{Q}} g \iff f|_{\mathbb{R}} g \iff f|_{\mathbb{C}} g.$$