The Quadratic Formula

Theorem 8 (Quadratic Formula). Suppose $2 \neq 0$ in F, i.e., suppose $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or $\mathbb{F}_p$, where $p \neq 2$. If $f(x) = x^2 + bx + c \in F[x]$, then

$$f(x) \text{ is reducible in } F[x] \;\Leftrightarrow\; b^2 - 4c =: d^2$$

is a square in F. Moreover, in this case f factors as

$$f(x) = (x - \alpha_1)(x - \alpha_2),$$

where $\alpha_1 = \frac{1}{2}(-b + d)$ and $\alpha_2 = \frac{1}{2}(-b - d)$. In particular, $f(x)$ has the roots α_1 and α_2.

Remarks 1) The quadratic formula was already known (in essence) to the Babylonians in 1900BC!

2) The expression $\Delta(f) = b^2 - 4c$ is called the discriminant of $f(x) = x^2 + bx + c$.

Corollary 1. $f(x) = x^2 + bx + c$ is irreducible in $\mathbb{R}[x]$ $\Leftrightarrow \Delta(f) = b^2 - 4c < 0$.

Corollary 2 a) $f(x) = x^2 + bx + c \in \mathbb{F}_2[x]$ is irreducible $\Leftrightarrow f(x) = x^2 + x + 1$.

b) $f(x) = x^2 + bx + c \in \mathbb{F}_3[x]$ is irreducible $\Leftrightarrow$ $b^2 - 4c = 2$ (in $\mathbb{F}_3$) $\Leftrightarrow c = b^2 + 1$ (in $\mathbb{F}_3$).