The Multiplicity of a Root

Definition: The multiplicity of $a \in F$ as a root of $f(x) \in F[x]$ is

$$\text{mult}_a(f) = \text{expt}_{x-a}(f(x)).$$

Thus:

$$\text{mult}_a(f) \geq m \iff (x-a)^m \mid f(x).$$

Theorem 10 (Derivative Test). Let $f(x) \in \mathbb{C}[x]$ be a polynomial and $a \in \mathbb{C}$. Then

$$\text{mult}_a(f) \geq m \Leftrightarrow f(a) = f'(a) = \ldots = f^{(m-1)} = 0,$$

where f', $f'' = f^{(2)}, \ldots, f^{(k)}, \ldots$ denote the 1^{st}, 2^{nd}, $\ldots$, k-th, $\ldots$ derivatives of mcf. In particular,

$$\text{mult}_a(f) = m \Leftrightarrow$$
$$f(a) = f'(a) = \ldots = f^{(m-1)} = 0 \text{ and } f^{(m)} \neq 0.$$

Example: Let $f(x) = (x-1)^2(x-2)$. Then

$$\begin{aligned} f'(x) &= (x-1)(3x-5), \\ f''(x) &= (3x-5) + 3(x-1) = 6x - 8. \end{aligned}$$

Thus, $f(1) = f'(1) = 0$ but $f''(1) = -3 \neq 0$, so $\text{mult}_1(f) = 2$.