The Method of Comparing Coefficients

Observation: If

$$g = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_0,$$
$$h = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0,$$

then

$$f := g \cdot h = c_{m+n}x^{m+n} + c_{m+n-1}x^{m+n-1} + \ldots + c_0,$$

where the coefficients c_i of f are given by

$$c_i = a_0 b_i + a_1 b_{i-1} + \ldots + a_i b_0.$$

In particular:

$$c_0 = a_0 b_0 \quad \text{and} \quad c_{m+n} = a_m b_n.$$

Thus: if $g|f$ in $\mathbb{Z}[x]$, then

constant coeff. of g | constant coeff. of f

leading coeff. of g | leading coeff. of f

Note: Gauss's Lemma + Observation

$\Rightarrow$ Rational Root Test

Example: Factor $f(x) = x^4 - 4x^3 + 5x^2 - 4x + 1$.

1) Check for roots: By RRT (= Rational Root Test), only the roots ± 1 are possible, but

$$f(\pm 1) = 1^4 - 4(\pm 1) + 5 - 4(\pm 1) + 1 = 7 \pm 8 \neq 0$$

$\Rightarrow$ $f(x)$ has no linear factors

$\Rightarrow$ either: $f(x)$ is irreducible

or: $f(x) = g_1(x)g_2(x)$, g_i irred. quadratics

2) Try the factorization: $f(x) = g_1(x)g_2(x)$, where the $g_i(x)$ are quadratic. Note that we can choose $g_i \in \mathbb{Z}[x]$ by Gauss's Lemma; moreover, by the above Observation, g_i can be chosen to be monic:

$$g_i(x) = x^2 + a_i x + b_i, \text{ with } a_i, b_i \in \mathbb{Z}.$$

In addition: $b_1 b_2 = 1 \Rightarrow b_1 = b_2 = 1$ or $b_1 = b_2 = -1$

Thus, comparing the coefficients of f and $g_1 g_2$:

(1) $a_1 + a_2 = -4$

(2) $b_1 + a_1 a_2 + b_2 = 5$

(3) $a_1 b_2 + a_2 b_1 = -4$

from which we get $b_1 = b_2 = 1, a_1 = -1, a_2 = -3$, so

$$f(x) = (x^2 - x + 1)(x^2 - 3x + 1).$$