The Fundamental Theorem of Algebra

Theorem A: If $f(x)$ is a polynomial of positive degree (i.e. $\deg(f) > 0$), then f has a root in $\mathbb{C}$: there exists $z \in \mathbb{C}$ such that $f(z) = 0$.

Remarks: 1) The first (correct) proof of this result was given by C.F. Gauss in 1797 (published 1799) when he was 20 (22) years old. He gave two more proofs in 1816, and a fourth in 1849.

2) Earlier proofs given by d'Alembert, Euler, Fontenex, Lagrange were criticized by Gauss.

3) The proofs are not easy; all use analysis.

Corollary 1: The only monic irreducible polynomials in $\mathbb{C}[x]$ are the linear polynomials $x - a$, ($a \in \mathbb{C}$).

Corollary 2 ("Factorization Theorem in $\mathbb{C}[x]$"): Every polynomial $f(x) \in \mathbb{C}[x]$ of postive degree has the factorization

$$f(x) = c\,(x - a_1)^{n_1}(x - a_2)^{n_2} \cdots (x - a_r)^{n_r}.$$

Thus: "Every polynomial $f(x) \in \mathbb{C}[x]$ of degree n has precisely n roots in $\mathbb{C}$, if we count the roots according to their multiplicities."

Warning: The above Theorem A is not constructive because it doesn't give us a procedure for finding the root z. Similarly, Corollary 2 is not constructive.

In fact: If $n = \deg(f) \geq 5$, then there is no general formula for expressing the roots of $f(x)$ in terms of the coefficients of $f(x)$, using only the operations of $\pm$, $\cdot$, $\div$ and $\sqrt[m]{\cdot}$, for any $m \geq 2$.

This was first proven by:

Paolo Ruffini (1765–1822) in 1799 for $n = 5$ (but his proof contained a gap)

Niels Abel (1802–1829) in 1824 and 1826

Evariste Galois (1811–1832) in 1832.

Note: Galois developed a method for determining which equations $f(x) = 0$ can be solved by radicals and which cannot. This theory is now known as Galois Theory.

For example, by using his theory one can show that the roots of the quintic polynomial

$$f(x) = x^5 - 4x + 2$$

cannot be expressed in terms of radicals.