The Factorization Theorem in $\mathbb{R}[\mathbf{x}]$

Theorem B: A monic polynomial $p(x) \in \mathbb{R}[x]$ is irreducible over $\mathbb{R}$ if and only if:

either $p(x) = x - a$ is linear polynomial

or $p(x) = x^2 + bx + c$ is a quadratic polynomial with $b^2 < 4c$.

Corollary. (Factorization Theorem in $\mathbb{R}[x]$)

If $f(x)$ is a real polynomial, then its factorization in $\mathbb{R}[x]$ into monic irreducible factors has the the form

$$f(x) = c(x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_r)^{n_r} \cdot(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_sx+c_s)^{m_s},$$

where $c, a_1, \ldots, a_r, b_1, \ldots, b_s, c_1, \ldots, c_s \in \mathbb{R}$ are real numbers and $b_k^2 < 4c_k$, for $1 \leq k \leq s$.

Note. If, in the above factorization, we write

$$x^2 + b_kx + c_k = (x-\alpha_k)(x-\overline{\alpha_k}), \text{ where } \alpha_k \in \mathbb{C},$$

then the factorization of $f(x)$ in $\mathbb{C}[x]$ is given by:

$$f(x) = c(x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_r)^{n_r} \cdot(x-\alpha_1)^{m_1}(x-\overline{\alpha_1})^{m_1}\cdots \cdots(x-\alpha_s)^{m_s}(x-\overline{\alpha_s})^{m_s}.$$