The Binomial Theorem

Theorem 7 (Binomial Theorem) For any positive integer n we have

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots$$
$$\cdots + \binom{n}{n-1} x y^{n-1} + y^n,$$

where the binomial coefficients $\binom{n}{k} \in \mathbb{N}$ are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1}.$$

Thus $\binom{n}{0} = 1$ and $\binom{n}{1} = n$. For $k \geq 1$, they may be computed recursively by the formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Remark. A convenient way to generate the binomial coefficients for small values of n is by *Pascal's triangle*:

n	$\binom{n}{k}$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 $\underbrace{35\quad 35}$ 21 7 1
8	1 8 28 56 70 56 28 8 1

Historical Remark. Pascal's triangle is named after Blaise Pascal (1623–1662) because he used it ingeniously in his studies of probability. In fact, Pascal's triangle was already known to the Chinese around 1300 and also appeared on the title page of a book published in Europe in 1530.

Theorem 8. If p is a prime, then

$$\binom{p}{k} \equiv 0 \pmod{p} \quad \text{for } 1 \le k \le p-1.$$