Complex Numbers

Set of complex no's: $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$.

If $z = x+iy, x, y \in \mathbb{R}$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$ are called the real and imaginary parts of z.

Addition: Componentwise.

Multiplication: Use the fact that $i^2 = -1$:

$$(x + iy)(a + ib) = (xa - yb) + i(ya + xb).$$

Complex Conjugate: $\overline{z} = x - iy$, if $z = x + iy$.

0) $\overline{z} = z \;\Leftrightarrow\; z \in \mathbb{R}$

1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

2) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

3) $\overline{z^n} = \overline{z}^n$, for all $n = 1, 2, \ldots$.

4) $\overline{\overline{z}} = z$.

Absolute Value: $|z| = \sqrt{x^2 + y^2}$, if $z = x + iy$.

0) $|z|^2 = z\overline{z}$.

1) $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$.

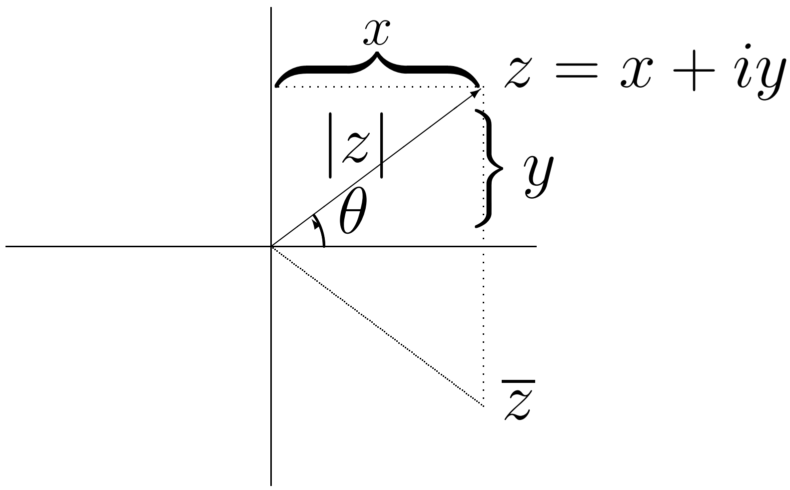
2) $|z_1 z_2| = |z_1| \cdot |z_2|$.

3) $|z_1 + z_2| \leq |z_1| + |z_2|$.

Inverses and Division: Use complex conjugate:

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{|z_2|^2}.$$

Geometrical Representation:



Polar Coordinates: (r, θ), where

$$\begin{aligned} r &= \sqrt{x^2+y^2} = |z| && \text{(absolute value)} \\ \theta &= \arg(z) && \text{(argument)} \\ &= \text{the angle in the diagram}, && (0 \le \theta < 2\pi). \end{aligned}$$

Moreover, by trigonometry we have

$$x = r \cos\ \theta, \quad y = r \sin\ \theta,$$

which yields the polar form representation of z:

$$z = |z|(\cos\ \theta + i \sin\ \theta).$$

The Multiplication Rule: We have

$$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

whenever $\begin{cases} z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \\ z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)) \end{cases}$.

Thus, to multiply two complex numbers, multiply their absolute values and add their arguments.

Powers (De Moivre's Formula): We have

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)), \quad \text{for all } n \geq 1,$$

and any $z = r(\cos(\theta) + i\sin(\theta))$.

The n-th Roots of $a = r(\cos(\alpha) + i\sin(\alpha))$ are:

$$z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\alpha+2\pi k}{n}\right) + i\sin\left(\frac{\alpha+2\pi k}{n}\right)\right),$$

where $0 \leq k < n$. Thus there are n such roots which are spaced equidistantly on the circle of radius $r^{\frac{1}{n}}$.

Exponentials and Euler's Formula: The function

$$e^z = e^x(\cos(y) + i\sin(y)), \quad z = x + iy,$$

satisfies the exponential law: $e^{z_1+z_2} = e^{z_1}e^{z_2}$. Thus

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \quad \text{(Euler's Formula)}$$