Solutions of Polynomial Equations

Basic Problem: Given a polynomial $f(x) \in \mathbb{Q}[x]$, find (or describe) all the solutions of $f(x) = 0$.

In this chapter we learned the following facts:

1. By the Fundamental Theorem of Algebra, all the solutions exist theoretically in $\mathbb{C}$. If we count them according to their multiplicities, then there are precisely $n = \deg(f)$ such solutions.

2. In general, the solutions cannot be described by radicals when $\deg(f) \geq 5$. (Theorem of Ruffini, Abel and Galois).

3. We can always factor $f(x)$ into its irreducible factors in $\mathbb{Q}[x]$, and there is a (complicated) algorithm for doing this (which we didn't learn). Then the rational solutions correspond to the linear factors.

   Each of the other solutions can be described by its associated irreducible polynomial $p(x) \in \mathbb{Q}[x]$. (This is not a good description, but it is the best that we can do in general.)

We also learned the following more general facts:

1. For any field F, where $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{F}_p$, a nonzero polynomial $f \in F[x]$ has at most $n = \deg(f)$ roots in F. Moreover, by the Factor Theorem there is a one to one correspondence between:

$$\{\text{roots } \alpha \text{ of } f\} \leftrightarrow \{\text{linear factors } x - \alpha \text{ of } f\}$$

2. For any field F, we can factor $f \in F[x]$ uniquely into a product of a constant and monic irreducible polynomials (Unique Factorization Theorem). If f has $n = \deg(f)$ distinct roots $a_1, \ldots, a_n$ in F, then there exists $c \in F$ such that

$$f(x) \;=\; c(x - a_1)(x - a_2) \cdots (x - a_n).$$

3. If $F = \mathbb{C}$, then all irreducible polynomials are linear (Fundamental Theorem of Algebra).

4. If $F = \mathbb{R}$, then every irreducible polynomial $p(x)$ has degree ≤ 2. (This follows from the previous assertion.)