Consider a general system of two linear equations in two unknowns:

$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

By eliminating variables we obtained the unique solution

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}}$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}.$$

provided that the expression $a_{11}a_{22} - a_{21}a_{12}$ in the denominators is non-zero.

Motivated by this we define the determinant of a 2×2 matrix as follows:

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\det A = a_{11}a_{22} - a_{21}a_{12}$.

How to remember the determinant of a 3×3 matrix.

Motivated by the solution obtained by elimination of variables for a general system of 3 equations in three unknowns we define

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

to be $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$.

This expression can be remembered in the following way. Write down the matrix, repeating the first two columns:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Then the terms with positive sign are the products of the downward pointing diagonals and the terms with negative sign are the product of the upward pointing diagonals.

For example $a_{11}a_{22}a_{33}$ is the product of the first downward diagonal and has a + sign, and $a_{32}a_{23}a_{11}$ is the second upward diagonal and has a negative sign.

The above description of the determinant of a 3×3 matrix A is competely ad hoc. In order to generalize to larger matrices we need a more conceptual approach:

Group the terms in $\det A$ as follows

$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} =$

$$a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) =$$

$$a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

For any square matrix *A*, let A_{ij} be the matrix obtained by crossing out the i^{th} row and the j^{th} column. Then $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$, $A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$, $A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Then the above expression for $\det A$ becomes

$$\det(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}.$$

Motivated by this example we define the determinant of an arbitrary square matrix recursively as follows (starting with the determinant of a 2×2 matrix defined as before): $\det A =$

$$a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots (-1)^{n+1} a_{1n} \det A_{1n} = \Sigma_{i=1}^{n} (-1)^{i+1} a_{1i} \det A_{1i}.$$
