We defined the determinant of an arbitrary square matrix *A* follows (starting with the determinant of a 1×1 or 2×2 matrix): $\det A =$

$$a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots (-1)^{n+1} a_{1n} \det A_{1n} = \Sigma_{i=1}^{n} (-1)^{i+1} a_{1i} \det A_{1i}.$$

It follows from this definition that the determinant of *A* is the sum of all possible products of entries of *A*, one from each row and one from each column. If *n*>1 half of these terms have a + sign and half have a - sign. There are *n*! (*n*-factorial) terms, where $n! = 1 \cdot 2 \cdot 3 \ldots \cdot n$. The factorial grows large rapidly, for example $3! = 6, 4! = 24, 5! = 120 \ldots 10! = 3628800, 30! = 2.65 \times 10^{32}$. Thus the definition is only useful for relatively small matrices.

Our definition of the determinant can be thought of as ``expansion along the first row". However there is really nothing special about the first row. One can expand along any row or column. The notation can be simplified a bit by the following notation:

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

where as before A_{ij} is the matrix obtained from *A* by deleting the i^{th} row and j^{th} column. C_{ij} is called the (*i*,*j*) cofactor of *A*. We then have the following:

Theorem 1. (page 182) The determinant of any $n \times n$ matrix *A* can be computed by cofactor expansion along any row or column. Along the i^{th} row the expansion is

$$\det A = \Sigma_{k=1}^{n} a_{ik} C_{ik}$$

and along the j^{th} column the expansion is

$$\det A = \Sigma_{k=1}^{n} a_{kj} C_{kj}.$$

(That is, we multiply each entry of the row or column by its cofactor and add.)

Our definition of $\det A$ was expansion along the first row.

In Theorem 1 the signs have been absorbed into the cofactor notation. The pattern of signs is

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ & & \cdots & \end{bmatrix}$$