Why determinants?

The purpose of determinants is to capture in one number the essential features of a matrix (or of the corresponding linear map).

Some of the key properties of determinants (of $n \times n$ matrices *A* and *B*) are

1. $\det(I_n) = 1$
2. *A* is invertible $\Leftrightarrow \det A \neq 0$.
3. $\det(AB) = \det(A)\det(B)$
4. If *A* is 2×2 then the linear mapping $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ multiplies areas by $\det A$. Similarly if *A* is 3×3 then the linear mapping $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ multiplies volumes by $\det A$.
5. Determinants can be used to give explicit formulas for the solution of a system of *n* equations in *n* unknowns, and for the inverse of an invertible matrix. They can also be used to give formulas for the area/volume of certain geometric figures.

We have $\det 0 = 0$ however we do not have $\det(A+B) = \det A + \det B$ (for all *A* and *B*). Determinants do a better job of capturing the multiplicative properties of matrices than the additive properties.

We computed the determinant of the following matrix by cofactor expansion along the first row:

$$A = \begin{bmatrix} 2 & -4 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$$

We obtained value 2. In a similar manner, expanding along the second column we obtain:

$$\det(A) = -(-4)\det\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} + 2\det\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} - 4\det\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

=4(-3-1)+2(-2-3)-4(2-9) =4(-4)+2(-5)-4(-7)=-16-10+28=2.

We use the rule of signs

$$A = \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

It follows easily from the cofactor expansion that

1. The determinant of an upper or lower triangular matrix is the product of the diagonal entries:

   (a)

$$\det\begin{bmatrix} 1 & 0 & 0 \\ 7 & 2 & 0 \\ 8 & 9 & -4 \end{bmatrix} = (1)(2)(-4) = -8$$

   (b)

$$\det\begin{bmatrix} 4 & 2 & -1 & 1 \\ 0 & 5 & -4 & -7 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 7 \end{bmatrix} = (4)(5)(-3)(7) = -420$$

2. If all entries in a row or column are 0, then the determinant is 0. Just expand along that row or column. For example

$$\det\begin{bmatrix}1 & 2 & 3\\0 & 0 & 0\\8 & 9 & -4\end{bmatrix}=0$$

and

$$\det\begin{bmatrix}1 & 0 & 3\\7 & 0 & 2\\8 & 0 & -4\end{bmatrix}=0$$

The computation of determinants by the cofactor expansion is not practical except for small matrices, because there are too many terms. The following makes the computation of determinants much more manageable:

Theorem 3. Page 187 Let *A* be a square matrix. Then

(a) If *B* is obtained from *A* by replacing R_i by $R_i + aR_j (i \neq j)$ then $\det B = \det A$.

(b) If *B* is obtained from *A* by interchanging two rows the $\det B = -\det A.$

(c) If *B* is obtained from *A* by replacing R_i by kR_i then $\det B = k \det A$.

Column operations have the same effect.