# Review of Linear Independence

**Definition:** A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$ is called a linear independent set if

(1) $$c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0},\ c_1, \ldots, c_k \in \mathbb{R}$$
$$\Rightarrow c_1 = c_2 = \ldots = c_k = 0.$$

**Note:** Linear independence can be checked by row reduction. More precisely, if

$A = (\vec{v}_1|\vec{v}_2|\ldots|v_k)$ is the associated $n \times k$ matrix,

then condition (1) holds if and only if

(2) $$A\vec{w} = \vec{0} \;\Rightarrow\; \vec{w} = \vec{0},\ \text{for all } \vec{w} \in \mathbb{R}^k$$

because $A(c_1, \ldots, c_k)^t = c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$. By row reduction, (2) holds if and only if

(3) $$\text{rank}(A) \;=\; k,$$

where, if $R$ is a row echelon form (REF) of $A$,

$$\text{rank}(A) \stackrel{\text{defn}}{=} \#\text{ of nonzero rows of } R$$
$$= \#\text{ of leading 1's in } R.$$

**Refinement:** As before, let $A = (\vec{v}_1|\vec{v}_2|\ldots|v_k)$ be an $n \times k$ matrix, and let $R$ be a REF of $A$. Put

$$I = \{i : \text{column } i \text{ of } R \text{ contains a leading } 1\}$$

so that (by definition) $\text{rank}(A) = \#I$.

Then we have:

1) the set $\{\vec{v}_i\}_{i\in I}$ is a linear independent set;

2) the set $\{\vec{v}_i\}_{i\in I}$ spans the space $V = \langle \vec{v}_1, \ldots \vec{v}_k \rangle$.

Thus, $\{\vec{v}_i\}_{i\in I}$ is a basis of $V$ and hence

$$\dim V = \operatorname{rank}(A) = \#I.$$

**Example.** Find a basis of $V = \langle \vec{v}_1, \ldots, \vec{v}_4 \rangle$, where $\vec{v}_1 = (1,2,3,4,5), \vec{v}_2 = (1,2,2,2,2)$, $\vec{v}_3 = (0,0,1,2,3), \vec{v}_4 = (1,1,1,1,1) \in \mathbb{R}^5$.

**Solution.** Put $A = (\vec{v}_1|\vec{v}_2|\vec{v}_3|\vec{v}_4)$. By row reduction we obtain

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 2 & 2 & 1 \\ 5 & 2 & 3 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

and so the leading $1$'s of $R$ (REF) are in columns $1, 2$ and $4$. Thus, $I = \{1,2,4\}$ and hence by 1), 2):

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_4\} \quad \text{is a basis of } V.$$

In particular, $\dim V = \operatorname{rank}(A) = \#I = 3$, so $V$ is 3-dimensional.

**Note:** If $n = k$, then the linear independence of the vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is equivalent to several other properties of the $n \times n$ matrix $A = (\vec{v}_1 | \cdots | \vec{v}_n)$, as the following basic fact from Linear Algebra shows:

**Fact:** If $A$ is an $n \times n$ matrix, then the following properties are equivalent:

(i) $A$ is invertible, i.e., there is a matrix $B$ such that $AB = BA = I$.

(ii) The columns of $A$ are linearly independent.

(iii) $A\vec{x} = \vec{0} \Rightarrow \vec{x} = 0$.

(iv) $A\vec{x} = \vec{y}$ has a solution $\vec{x}$ for every $\vec{y} \in \mathbb{R}^n$.

(iv′) $A\vec{x} = \vec{y}$ has a unique solution for every $\vec{y} \in \mathbb{R}^n$.

(v) $\mathrm{rank}(A) = n$.

(vi) $\det(A) \neq 0$.

Proof Sketch:

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v): See the above discussion.

(i) $\Rightarrow$ (iv)′ $\Rightarrow$ (iv): Take $\vec{x} = B\vec{y}$.

(iv) $\Rightarrow$ (i): Take $B = (\vec{v}_1 | \ldots | \vec{v}_n)$, where $A\vec{v}_i = \vec{e}_i$.

(i) $\Rightarrow$ (vi): Since $\det(A)\det(B) = \det(AB) = \det(I) = 1$, we have that $\det(A) \neq 0$.

(vi) $\Rightarrow$ (i): Cofactor formula for $B = A^{-1}$.