The Gram-Schmidt Orthogonalization Procedure

Theorem 10 (Gram-Schmidt): Let $\vec{v}_1, \ldots, \vec{v}_k$ be any basis of $V \subset \mathbb{R}^n$, and put

$$\begin{aligned}
\vec{b}_1 &= \vec{v}_1 \\
\vec{b}_2 &= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1}\vec{b}_1 &&= \vec{v}_2 - P_{V_1}(\vec{v}_2) \\
\vdots \quad &\vdots \qquad\qquad \vdots && \vdots \qquad\qquad \vdots \\
\vec{b}_k &= \vec{v}_k - \sum_{i=1}^{k-1} \frac{\vec{v}_k \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}\vec{b}_i &&= \vec{v}_k - P_{V_{k-1}}(\vec{v}_k),
\end{aligned}$$

where $V_i = \langle \vec{v}_1, \ldots, \vec{v}_i \rangle = \langle \vec{b}_1, \ldots, \vec{b}_i \rangle$, if $1 \le i \le k$.

Then:

(a) $\vec{b}_1, \ldots, \vec{b}_k$ is an orthogonal basis of V, and

(b) $\frac{\vec{b}_1}{\|\vec{b}_1\|}, \ldots, \frac{\vec{b}_k}{\|\vec{b}_k\|}$ is an orthonormal basis of V.