Finding rem(f,g): an Example

Example: Let $g(t) = (t-1)^2(t-2)$. Find a formula for $\text{rem}(f, g)$ which is valid for every $f \in \mathbb{C}[t]$.

Solution: Write $r(t) = \text{rem}(f, g) = a_0 + a_1 t + a_2 t^2$. Then the a_i's satisfy the following system of (linear) equations:

$$\begin{array}{rcl|rcl} r(1) & = & f(1) & a_0 + a_1 + a_2 & = & f(1) \\ r'(1) & = & f'(1) & a_1 + 2a_2 & = & f'(1) \\ r(2) & = & f(2) & a_0 + 2a_1 + 4a_2 & = & f(2) \end{array}$$

Row-reducing the associated augmented matrix yields:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & f(1) \\ 0 & 1 & 2 & f'(1) \\ 1 & 2 & 4 & f(2) \end{array}\right) \overset{R3-R1}{\leftrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & f(1) \\ 0 & 1 & 2 & f'(1) \\ 0 & 1 & 3 & f(2) - f(1) \end{array}\right)$$

$$\overset{R3-R2}{\leftrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & f(1) \\ 0 & 1 & 2 & f'(1) \\ 0 & 0 & 1 & f(2) - f(1) - f'(1) \end{array}\right).$$

Thus, by back-substitution we obtain:

$$\begin{aligned} a_2 &= f(2) - f(1) - f'(1) \\ a_1 &= \quad f'(1) - 2a_2 &&= f'(1) - 2f(2) + 2f(1) + 2f'(1) \\ & &&= 3f'(1) - 2f(2) + 2f(1), \\ a_0 &= f(1) - a_1 - a_2 &&= f(1) - 3f'(1) + 2f(2) - 2f(1) \\ & &&\quad -f(2) + f(1) + f'(1) \\ & &&= -2f'(1) + f(2), \end{aligned}$$

and so we have

$$(1)\quad r(t) = (f(2) - 2f'(1)) + (2f(1) + 3f'(1) - 2f(2))t + (f(2) - f(1) - f'(1))t^2.$$

Note: The above formula exhibits the coefficients of $r(t)$ explicitly. However, it is frequently useful to re-write the above formula in a way that exhibits its dependence on f more clearly.

Example: If $g(t) = (t-1)^2(t-2)$, find a formula for $\text{rem}(f,g)$ in terms of $f(1)$, $f'(1)$ and $f(2)$.

Solution: Looking at the result (1) of the previous example, we shall separate out and collect the terms involving $f(1)$, $f'(1)$ and $f(2)$:

$$\begin{aligned} r(t) &\overset{(1)}{=} (f(2) - 2f'(1)) + (2f(1) + 3f'(1) - 2f(2))t \\ &\quad +(f(2) - f(1) - f'(1))t^2 \\ &= f(1)(2t - t^2) + f'(1)(-2 + 3t - t^2) \\ &\quad +f(2)(1 - 2t + t^2). \end{aligned}$$

In other words, we have for every $f \in \mathbb{C}[t]$:

$$\text{rem}(f,g) = f(1)e_{10}(t) + f'(1)e_{11}(t) + f(2)e_{20}(t)$$

where

$$\begin{aligned} e_{10}(t) &= 2t - t^2 &&= t(2-t) \\ e_{11}(t) &= -2 + 3t - t^2 &&= (1-t)(t-2) \\ e_{20}(t) &= 1 - 2t + t^2 &&= (1-t)^2. \end{aligned}$$

Note that these polynomials do not depend on f.