The Direct Sum of Subspaces

Definition: (a) If $\vec{v} = (v_1, \ldots, v_n)^t \in \mathbb{C}^n$ and $\vec{w} = (w_1, \ldots, w_m)^t \in \mathbb{C}^m$, then the vector

$$\vec{v} \oplus \vec{w} := (v_1, \ldots, v_n, w_1, \ldots, w_m)^t \in \mathbb{C}^{n+m}$$

is called the *direct sum* of $\vec{v}$ and $\vec{w}$.

(b) If $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ are subspaces, then the *direct sum* of V and W is the subspace of $\mathbb{C}^{n+m}$ defined by

$$V \oplus W = \{\vec{v} \oplus \vec{w} \in \mathbb{C}^{n+m} : \vec{v} \in V, \vec{w} \in W\}.$$

Remarks: 1) If $\vec{v}_1, \ldots, \vec{v}_r$ is a basis of $V \subset \mathbb{C}^n$ and $\vec{w}_1, \ldots, \vec{w}_s$ is one of $W \subset \mathbb{C}^m$, then

$$\vec{v}_1 \oplus \vec{0}_m, \ \ldots, \ \vec{v}_r \oplus \vec{0}_m, \ \vec{0}_n \oplus \vec{w}_1, \ \ldots, \ \vec{0}_n \oplus \vec{w}_s$$

is a basis of $V \oplus W$. (Here, $\vec{0}_m = (\underbrace{0, \ldots, 0}_{m})^t$.) Thus

$$\dim(V \oplus W) = \dim V + \dim W. \tag{1}$$

2) If A is an $a \times m$ matrix and B is a $b \times n$ matrix, then for every $\vec{v} \in \mathbb{C}^m$ and $\vec{w} \in \mathbb{C}^n$ we have

$$\operatorname{diag}(A, B)(\vec{v} \oplus \vec{w}) = (A\vec{v}) \oplus (B\vec{w}). \tag{2}$$