Properties of Constituent Matrices

Notation: If A is an $m \times m$ matrix, let

$$\mathrm{ch}_A(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_s)^{m_s}$$

be the factorization of $\mathrm{ch}_A(t)$ (λ_i's distinct), and let $\{E_{ik}\}$ be the set of constituent matrices of A.

For any $\lambda \in \mathbb{C}$ and $k \geq 0$ put

$$E^A_{\lambda,k} = \begin{cases} E_{ik} & \text{if } \lambda = \lambda_i \text{ and } k \leq m_i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem: Let $\lambda \in \mathbb{C}$ and $k \geq 0$. Then:

(a) If $A = \mathrm{diag}(A_1, \ldots, A_r)$, then

(1) $$E^A_{\lambda,k} = \mathrm{diag}(E^{A_1}_{\lambda,k}, \ldots E^{A_r}_{\lambda,k}).$$

(b) If $B = P^{-1}AP$, then

(2) $$E^B_{\lambda,k} = P^{-1}E^A_{\lambda,k}P.$$

Corollary 1: In the situation of part (a) we have

(3) $E^A_{\lambda,k} = 0$ for $k \geq \max(m_{A_1}(\lambda), \ldots, m_{A_r}(\lambda))$.

Application: The maximum size of a Jordan block.

Let $\lambda \in \mathbb{C}$, and put

$t_A(\lambda) := \max\{k : J(\lambda, k)$ appears as a block in $J_A\}$,

where J_A is the Jordan canonical form of A.

Corollary 2: We have that

$$E^A_{\lambda,k} = 0 \quad \Leftrightarrow \quad k \geq t_A(\lambda). \tag{4}$$

Remarks: 1) Recall that λ is a regular eigenvalue of $A \Leftrightarrow t_A(\lambda) = 1$. Thus by Corollary 2 we have

$$\lambda \text{ is regular eigenvalue} \Rightarrow E^A_{\lambda,k} = 0 \text{ for } k \geq 1.$$

2) The number $t_A(\lambda)$ is connected with the generalized geometric multiplicities $\nu^p_A(\lambda)$ as follows:

$$\nu^k_A(\lambda) = \nu^{k+1}_A(\lambda) \quad \Leftrightarrow \quad k \geq t_A(\lambda). \tag{5}$$

Thus, we have the following pattern (if $t = t_A(\lambda)$):

$$0 < \nu^1_A(\lambda) < \ldots < \nu^t_A(\lambda) = \nu^{t+1}_A(\lambda) = \ldots$$

3) If we put

$$\mu_A(t) = (t - \lambda_1)^{t_A(\lambda_1)} \ldots (t - \lambda_s)^{t_A(\lambda_s)},$$

then $\mu_A(t) | \mathrm{ch}_A(t)$ and we have $\mu_A(A) = 0$ by the Spectral Decomposition Formula and (4). This refines the Cayley-Hamilton Theorem.