The Lagrange Interpolation Formula

Problem: ("Exact Fit") Given the n "data points" $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, where the x_i's are distinct, find a polynomial of least degree such that

$$(\star) \qquad \left.\begin{array}{rcl} f(x_1) & = & y_1 \\ f(x_2) & = & y_2 \\ \vdots & \vdots & \vdots \\ f(x_n) & = & y_n \end{array}\right\} \quad \begin{array}{c}\text{"graph of } y = f(x) \\ \text{passes through} \\ (x_1, y_1), \ldots, (x_n, y_n)\text{"}\end{array}$$

Theorem 1: ("Lagrange Interpolation Formula") The unique polynomial $f(x)$ of degree $\leq n-1$ which passes through $(x_1, y_1), \ldots, (x_n, y_n)$ is given by the formula

$$f(x) = \sum_{k=1}^{n} y_k \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \sum_{k=1}^{n} y_k \frac{g_k(x)}{g_k(x_k)},$$

where

$$g_k(x) = (x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)$$

$$= \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x - x_k)}.$$

Note: There is a close connection between the Lagrange interpolation polynomial and remainders:

Theorem 2: Suppose

$$g(x) = (x - a_1)(x - a_2)\cdots(x - a_n),$$

where the a_i's are distinct. Then for any polynomial $f(x)$ we have

$$\text{rem}(f, g) = \sum_{k=1}^{n} f(a_k)e_k(x),$$

where

$$e_k(x) = \frac{g_k(x)}{g_k(a_k)} \quad \text{with} \quad g_k(x) = \frac{g(x)}{x - a_k}.$$

Thus, $\text{rem}(f, g)$ is the Lagrange interpolation polynomial of the data points $(a_1, f(a_1))$, $(a_2, f(a_2))$, $\ldots$, $(a_n, f(a_n))$.

Remarks. Note that the e_k's depend only on g (and not on f); we call these the constituent polynomials of g.