The Lagrange Interpolation Polynomial

Theorem 3 ("Matrix Method")

The coefficients $a_0, a_1, \ldots, a_{n-1}$ of the unique interpolation polynomial

$$f(x) = \sum_{k=0}^{n-1} a_k x^k$$

of degree $\leq n-1$ which passes through the data points $(x_1, y_1), \ldots, (x_n, y_n)$ is given by the following system of linear equations

$$\underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}}_{\vec{a}} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\vec{y}}.$$

Note: The above matrix A is called the Vandermonde matrix defined by $x_1, x_2, \ldots, x_n$. Its determinant is called the Vandermonde determinant:

$$V(x_1, x_2, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$