The Geometry of $\mathbb{R}^n$

(a) Points, vectors, angles:

$\vec{x} = (x_1, x_2, \ldots, x_n)$	a *point* (or *vector*) in $\mathbb{R}^n$,
$\vec{x} \cdot \vec{y} = x_1 y_1 + \ldots + x_n y_n$	the *dot product* of vectors $\vec{x}$ and $\vec{y}$,
$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}},$	the *length* (or *norm*) of $\vec{x}$,
$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$	the *distance* between $\vec{x}$ and $\vec{y}$,
$\cos(\vec{x}, \vec{y}) = \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \cdot \|\vec{y}\|}$	the *cosine of the angle* between two non-zero vectors $\vec{x}$ and $\vec{y}$.

Remark: To justify the name "$\cos(\vec{x}, \vec{y})$", we need:

Theorem 6 (Cauchy-Schwarz Inequality) For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \cdot \|\vec{y}\|.$$

Definition. The *angle* between two non-zero vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ is the unique number θ such that

$$\cos(\theta) \ = \ \cos(\vec{x}, \vec{y}), \quad \text{and} \quad 0 \leq \theta \leq \pi.$$

Corollary (Triangle Inequality). For any vectors $\vec{x}, \vec{y}$ and $\vec{z} \in \mathbb{R}^n$

$$d(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{z}) + d(\vec{y}, \vec{z}).$$

In particular, $\|\vec{x} - \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

Theorem 5 (Law of Cosines). For any $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\| \cdot \|\vec{y}\| \cos(\vec{x}, \vec{y})$$

Corollary (Rule of Pythagoras). If $\vec{x} \cdot \vec{y} = 0$ (i.e. if $\vec{x}$ is perpendicular to $\vec{y}$), then $\|\vec{x} - \vec{y}\|^2 \ = \ \|\vec{x}\|^2 + \|\vec{y}\|^2$.

(b) Equations of lines, planes, and linear sets:

Besides points, we shall also need to consider lines, planes and, more generally, linear sets in $\mathbb{R}^n$. These are defined as follows.

Line: through two points $\vec{a}, \vec{b} \in \mathbb{R}^n$:

$$L: \ \vec{x} = \vec{a} + (\vec{b} - \vec{a})t, \ \ t \in \mathbb{R}.$$

Plane: through the three points $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$:

$$S: \ \vec{x} = \vec{a} + (\vec{b} - \vec{a})t_1 + (\vec{c} - \vec{a})t_2, \ t_1, t_2 \in \mathbb{R}.$$

Linear set: through the $r+1$ points $\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_r \in \mathbb{R}^n$:

$$S: \ \vec{x} = \vec{a}_0 + (\vec{a}_1 - \vec{a}_0)t_1 + \ldots + (\vec{a}_r - \vec{a}_0)t_r, \ t_1, \ldots, t_r \in \mathbb{R}.$$

Linear space (= a linear set containing the origin $\vec{0}$): determined (or *spanned*) by the r points/vectors $\vec{a}_1, \ldots, \vec{a}_r \in \mathbb{R}^n$:

$$L = \langle \vec{a}_1, \ldots, \vec{a}_r \rangle = \{\vec{x} = \vec{a}_1 t_1 + \ldots + \vec{a}_r t_r : t_1, \ldots, t_r \in \mathbb{R}\}.$$

Note: We also say that L is the *linear span* of the vectors $\vec{a}_1, \ldots, \vec{a}_r$.

Remark. Linear sets often arise as follows. If

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \ldots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \ldots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & \vdots & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \ldots + & a_{mn}x_n & = & b_m \end{array}$$

is a system of m linear equations in n variables (which in matrix form may be written as $A\vec{x} = \vec{b}$), then its *solution set* $Sol(A\vec{x} = \vec{b})$ is a linear set in $\mathbb{R}^n$. To transform it into a vector equation, use row reduction and back-substitution.