Two Distance Problems

Problem A: Given an $n \times m$ matrix A and a vector $\vec{y} \in \mathbb{R}^n$, find a vector $\vec{w} \in \mathbb{R}^m$ such that

$$\|A\vec{w} - \vec{y}\| \quad \text{is minimal among all } \vec{w} \in \mathbb{R}^m.$$

Observation: If we represent A in the form

$$A = (\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_m)\,,$$

i.e. $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ denote the column vectors of A, then

$$V_A := \{A\vec{w} : \vec{w} \in \mathbb{R}^m\} = \langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \rangle$$

is the subspace of $\mathbb{R}^n$ spanned by the columns of A (called the column space of A).

Problem B: Given vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$, find a vector $\vec{v}_0 \in V := \langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \rangle$ such that

$$\|\vec{y} - \vec{v}_0\| \quad \text{is minimal among all } \vec{v}_0 \in V,$$

i.e. such that

$$d(\vec{y}, \vec{v}_0) \leq d(\vec{y}, \vec{v}), \quad \text{for all } \vec{v} \in V.$$