Solution of the Two Distance Problems

Theorem 7: (Geometry) If $\vec{v}_0 \in V$ is such that

(1) $$\vec{v} \cdot (\vec{y} - \vec{v}_0) = 0, \quad \text{for all } \vec{v} \in V,$$

(where $V = \langle \vec{v}_1, \ldots, \vec{v}_k \rangle$), then we have

(2) $$d(\vec{y}, \vec{v}_0) < d(\vec{y}, \vec{v}), \quad \text{for all } \vec{v} \in V, \vec{v} \neq \vec{v}_0,$$

and hence $P_V(\vec{y}) := \vec{v}_0$ is a solution to Problem B.

Theorem 8: Let $\vec{y} \in \mathbb{R}^n$ and $V = \langle \vec{v}_1, \ldots, \vec{v}_k \rangle \subset \mathbb{R}^n$ and put $A = (\vec{v}_1 | \ldots | \vec{v}_k)$. If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent, then A^tA is invertible, and the vector

(3) $$\vec{w}_0 := (A^tA)^{-1}A^t\vec{y},$$

has the property that

(4) $||A\vec{w}_0 - \vec{y}|| < ||A\vec{w} - \vec{y}||$, for all $\vec{w} \in \mathbb{R}^k$ with $\vec{w} \neq \vec{w}_0$. Moreover, $\vec{w}_0$ is the unique vector of $\mathbb{R}^k$ such that (4) holds, and hence it is the unique solution of Problem A. Thus $A\vec{w}_0$ is the unique solution of Problem B, i.e.,

(5) $$P_V(\vec{y}) = A\vec{w}_0 = \underbrace{A(A^tA)^{-1}A^t}_{P_V}\vec{y}$$

is the orthogonal projection of $\vec{y}$ onto V.