Orthogonal Projection Examples

Example 1: Find the orthogonal projection of $\vec{y} = (2,3)$ onto the line $L = \langle (3,1) \rangle$.

Solution: Let $A = (3,1)^t$. By Theorem 4.8, the orthogonal projection is given by

$$\begin{aligned} P_V(\vec{y}) &= A(A^tA)^{-1}A^t\vec{y} \\ &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \left((3,1) \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right)^{-1} \left((3,1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} ((10))^{-1} (9) = \frac{9}{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \end{aligned}$$

Example 2: Let $V = \langle (1,0,1), (1,1,0) \rangle$. Find the vector $\vec{v} \in V$ which is closest to $\vec{y} = (1,2,3)$.

Solution: By Theorem 4.7, the desired vector is the orthogonal projection $\vec{v} = P_V(\vec{y})$. Put $A = (\vec{v}_1|\vec{v}_2)$, where $\vec{v}_1 = (1,0,1)$ and $\vec{v}_2 = (1,1,0)$. Since the columns of A are a basis of V, Theorem 4.8 tells us that $P_V(\vec{y}) = A(A^tA)^{-1}A^t\vec{y}$. Now

$$A^tA = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and hence

$$(A^tA)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Moreover,

$$A^t\vec{y} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Thus, the closest vector $\vec{v} = P_V(\vec{y})$ is given by

$$\begin{aligned} P_V(\vec{y}) &= A(A^tA)^{-1}A^t\vec{y} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}. \end{aligned}$$