Orthogonal Vectors and Orthogonal Projection

Definition: A set $\vec{v}_1, \ldots, \vec{v}_k$ of vectors of a subspace $V \subset \mathbb{R}^n$ is called a *basis* of V if

(i) the vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent;

(ii) the vectors $\vec{v}_1, \ldots, \vec{v}_k$ span V, i.e., $V = \langle \vec{v}_1, \ldots, \vec{v}_k \rangle$.

Fact: Each subspace $V \subset \mathbb{R}^n$ has a basis, and all bases of V have the same number of elements, called the *dimension* of V.

Definition: A basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k\}$ of V is called an *orthogonal basis* if we have

$$\vec{b}_i \cdot \vec{b}_j = 0, \quad \text{for all } i \neq j. \tag{1}$$

It is called an *orthonormal basis* if it is an orthogonal basis which satisfies $||\vec{b}_i|| = 1$, for all i.

Note: If $\vec{b}_1, \ldots, \vec{b}_k$ is a set of non-zero vectors which satisfies (1), then they are linearly independent.

Theorem 9: Let $\vec{b}_1, \ldots, \vec{b}_k$ be an orthogonal basis of $V \subset \mathbb{R}^n$. Then the orthogonal projection of $\vec{y} \in \mathbb{R}^n$ onto V is given by:

$$P_V(\vec{y}) = \sum_{i=1}^{k} \frac{\vec{y} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i} \vec{b}_i.$$