Orthogonal Matrices

Definition: An $n \times n$ matrix $B = (\vec{b}_1|\vec{b}_2|\ldots|\vec{b}_n)$ is called orthogonal if its column vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ form an orthonormal basis of $\mathbb{R}^n$.

Theorem 11: If $B = (\vec{b}_1|\vec{b}_2|\ldots|\vec{b}_n)$ is an $n \times n$ matrix, then the following conditions are equivalent:

(1) B is orthogonal (i.e. the column vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ are an orthonormal basis of $\mathbb{R}^n$;

(2) $B^tB = I$, i.e. $B^{-1} = B^t$;

(3) $(B\vec{v} \cdot B\vec{w}) = (\vec{v} \cdot \vec{w})$, for all $\vec{v}, \vec{w} \in \mathbb{R}^n$;

(4) $||B\vec{v}|| = ||\vec{v}||$, for all $\vec{v} \in \mathbb{R}^n$.

Note: Thus, the orthogonal matrices are precisely those that preserve lengths and angles (when viewed as a linear transformation).