Fourier Approximation

Problem C: Given: 1) A function $g \in \mathcal{C}[a,b]$, and
2) "simple" functions $f_1, \ldots, f_k \in \mathcal{C}[a,b]$.

Find: a linear combination

$$f_0 \;=\; a_1 f_1 + \ldots + a_k f_k$$

which best approximates the function g in the L^2-norm; in other words, find f_0 as above such that

$$\int_a^b (f_0 - g)^2 dx \;\leq\; \int_a^b (f-g)^2,$$

for all functions $f = a_1' f_1 + \ldots + a_k' f_k \in V := \langle f_1, \ldots, f_k \rangle$.

Solution: Step 1: Apply the Gram-Schmidt Method to the functions $f_1, \ldots, f_k$ by defining the dot product of two functions $f, g \in \mathcal{C}[a,b]$ to be

$$(f \cdot g) = \int_a^b f(x)g(x)dx.$$

This gives us orthogonal functions $h_1, \ldots, h_k$ which are certain linear combinations of $f_1, \ldots, f_k$.

Step 2: Use the projection formula of Theorem 9:

$$f_0 = P_V(g) = \sum_{i=1}^{k} \frac{(g \cdot h_i)}{(h_i \cdot h_i)} h_i$$