Diagonalization Theorems

Theorem 3 (Diagonalization Theorem)

(a) An $m \times m$ matrix A is diagonable if and only if A has m linearly independent eigenvectors. of A.

(b) Suppose $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{C}^m$ is a linearly independent set of eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$ (so $A\vec{v}_i = \lambda_i \vec{v}_i$, for $1 \leq i \leq m$). Then the matrix $P = (\vec{v}_1 | \ldots | \vec{v}_m)$ is invertible and we have

$$P^{-1}AP = \mathrm{Diag}(\lambda_1, \ldots, \lambda_m).$$

Remark: Write

$$\mathrm{ch}_A(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_r)^{m_r},$$

where the λ_i's are distinct, and let B_i be a basis of the corresponding eigenspaces $E_A(\lambda_i)$, $1 \leq i \leq r$. Then it can be shown that

$$B = B_1 \cup B_2 \cup \ldots \cup B_r$$

is a linearly independent set. Thus:

$$A \text{ is diagonable} \Leftrightarrow B \text{ is a basis of } \mathbb{C}^m \Leftrightarrow \#B = m$$
$$\Leftrightarrow \dim E_A(\lambda_1) + \ldots + \dim E_A(\lambda_r) = m.$$

Corollary: If A has m distinct eigenvalues (i.e. if $\mathrm{ch}_A(t)$ has m distinct roots), then A is diagonable.

Warning: The converse of this corollary is false: a matrix A can be diagonable yet have repeated eigenvalues.

Example: the $m \times m$ identity matrix I is diagonal (hence diagonable), but has only one eigenvalue $\lambda_1 = 1$ (repeated m times).

Remark: If A is a real matrix (i.e. all entries of A are real numbers) and is symmetric, i.e. $A^t = A$, then A is automatically diagonable, as the the following useful result shows:

Theorem 4 (Principal Axis Theorem)

If A is a real symmetric matrix, then A is orthogonally diagonable; in other words, there exists an orthogonal matrix P (i.e. a real matrix satisfying $P^{-1} = P^t$) such that $P^{-1}AP$ is a diagonal matrix.

Remark: The name of this theorem comes from the fact that this theorem can used to show that quadrics in $\mathbb{R}^n$ centered at the origin (e.g. ellipses in $\mathbb{R}^2$, ellipsoids in $\mathbb{R}^3$, etc.) can be rotated so that their principal axes are along the coordinate axes of $\mathbb{R}^n$.