The Lagrange-Sylvester Interpolation Formula

Theorem 11 (Lagrange-Sylvester Interpol. Formula)

Let: $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ (distinct), $m_1, \ldots, m_s \in \mathbb{N}$,

$c_{ik} \in \mathbb{C}$, $1 \le i \le s$, $0 \le k \le m_i - 1$

be given, and put

$$h(t) = \sum_{i=1}^{s} \sum_{k=0}^{m_i-1} c_{ik} e_{ik}(t),$$

where the $e_{ik}(t)$ are the "constituent polynomials" of $g(t) := (t-\lambda_1)^{m_1} \cdots (t-\lambda_s)^{m_s}$. Then $\deg(h) < n := \deg(g)$, and $h(t)$ satisfies the relations

$$(1) \quad h^{(k)}(\lambda_i) = c_{ik}, \quad 1 \le i \le s, \; 0 \le k \le m_i - 1.$$

Moreover, $h(t)$ is the unique polynomial of degree $\le n-1$ satisfying these relations.

Remarks: 1) Clearly, Theorem 11 $\Rightarrow$ Theorem 10. (Take $c_{ik} = f^{(k)}(\lambda_i)$.)

2) It can be shown that the polynomials $e_{ik}(t)$ satisfy the relations

$$e_{ik}^{(l)}(\lambda_j) = \begin{cases} 1 & \text{if } i = j \text{ and } k = l \\ 0 & \text{otherwise} \end{cases}.$$

This can be used to prove explicit formulae for the $e_{ik}(t)$'s.