Discrete Linear Systems

Definition: A *discrete linear system* (d.l.s.) is a system of equations of the form

$$\vec{u}_{n+1} = A_n \vec{u}_n + \vec{b}_n$$

where $n = 0, 1, 2, \ldots$. Here the vectors $\vec{b}_0, \vec{b}_1, \ldots \in \mathbb{C}^m$ and the matrices A_n (all of size $m \times m$) are given, and we want to solve for the vectors $\vec{u}_n$ in terms of $\vec{u}_0$.

The system is said to have *constant coefficients* if $A_n = A$ is independent of n. Moreover, it is called *homogeneous* if $\vec{b}_n = \vec{0}$ for all n.

Theorem 13: The unique solution of a homogeneous discrete linear system $\vec{v}_{n+1} = A\vec{v}_n$ is given by

$$\vec{v}_n = A^n \vec{v}_0.$$

Corollary: If $\lambda_1, \ldots, \lambda_s$ are the distinct eigenvalues of A with multiplicities $m_1, \ldots, m_s$, then we have

$$\vec{v}_n = \sum_{i=1}^{s} \sum_{k=0}^{m_i - 1} k! \binom{n}{k} \lambda_i^{n-k} \vec{u}_{ik},$$

where the vectors $\vec{u}_{ik} = E_{ik} \vec{v}_0$ do not depend on n.