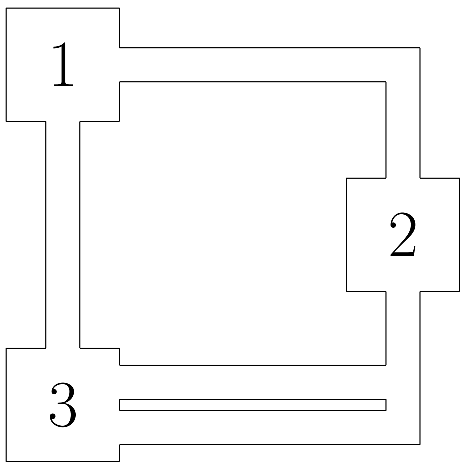
A Rat Maze

Consider the following system of three chambers connected by passages as shown.



At time $t = 0$, a rat is placed in one of the chambers (say in chamber 1).

Each minute thereafter, the rat is driven out of its present chamber by some stimulus and is prevented from re-entering immediately.

Assume: the rat chooses the exits of each chamber at random.

Question: What is the probability that the rat is in a certain chamber after n minutes?

Note: If s_i denotes the state that the rat is in chamber i, then we have a Markov chain with transition matrix

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Analysis: By definition, the ij-th entry a_{ij} of the transition matrix $A = (a_{ij})$ is the probability that the rat goes from chamber j to chamber i.

Now: If the rat ...

...is in chamber 1: chambers 2 and 3 are equally likely

$\Rightarrow$ the probability is $\frac{1}{2}, \frac{1}{2}$ each,

$\Rightarrow a_{21} = a_{31} = \frac{1}{2}$;

...is in chamber 2: chambers 1 and 3 are equally likely

$\Rightarrow$ the probability is $\frac{1}{2}, \frac{1}{2}$ each,

$\Rightarrow a_{12} = a_{32} = \frac{1}{2}$;

...is in chamber 3: there are 2 exits to chamber 2 and 1 exit to chamber 1

$\Rightarrow$ chamber 2 is twice as likely as chamber 1,

$\Rightarrow a_{13} = \frac{1}{3}, a_{23} = \frac{2}{3}$.

Thus, this is a Markov chain with transition matrix

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

More explicitly, if $v_{n,i}$ denotes the probability that the rat is in chamber i at time $t = n$, and if

$$\vec{v}_n \; = \; (v_{n,1}, v_{n,2}, v_{n,3})^t$$

is the associated probability distribution vector, then we have the Markov chain

$$\vec{v}_{n+1} \; = \; A\vec{v}_n, \quad \text{and so} \quad \vec{v}_n = A^n \vec{v}_0.$$

Since A has characteristic polynomial

$$\mathrm{ch}_A(t) \; = \; t^3 - \tfrac{3}{4}t - \tfrac{1}{4} \; = \; (t-1)(t+\tfrac{1}{2})^2,$$

the Spectral Decomposition Theorem shows that

$$\begin{aligned} A^n \; &= \; E_{10} + \left(-\tfrac{1}{2}\right)^n E_{20} + n\left(-\tfrac{1}{2}\right)^{n-1} E_{21} \\ &= \; E_{10} + \left(-\tfrac{1}{2}\right)^n (E_{20} - 2nE_{21}), \end{aligned}$$

where the constituent matrices E_{ik} are given by

$$E_{10} = \tfrac{1}{27}\begin{pmatrix} 8 & 8 & 8 \\ 10 & 10 & 10 \\ 9 & 9 & 9 \end{pmatrix}, \quad E_{20} = \tfrac{1}{27}\begin{pmatrix} 19 & -8 & -8 \\ -10 & 17 & -10 \\ -9 & -9 & 18 \end{pmatrix}, \quad E_{21} = \tfrac{1}{18}\begin{pmatrix} 1 & 1 & -2 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, if $\vec{v}_0 = (1,0,0)^t$, then

$$\vec{v}_n = \frac{1}{27}\left(\begin{pmatrix} 8 \\ 10 \\ 9 \end{pmatrix} + \left(-\tfrac{1}{2}\right)^n \begin{pmatrix} 19 - 3n \\ 3n - 10 \\ -9 \end{pmatrix}\right).$$